

Normal Forms for Perturbed Keplerian Systems¹

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Perturbations of rotating and nonrotating Keplerian systems with two and three degrees of freedom are considered. The dominant (unperturbed) part of the Hamilton function is the sum of the two-body part and the Coriolis term. The flows associated to these two components are used to put the whole system into normal form by means of symplectic transformations. Three different situations are analysed: (i) the two-body effect dominates the Coriolis part (slow rotations), (ii) the Coriolis effect is stronger than the two-body Hamiltonian (very fast rotations), and (iii) the two effects are comparable (moderate rotations). After performing the transformations to normal form, the system is reduced by one or two degrees of freedom. We describe the reduced phase spaces by calculating the invariants of the symmetry groups associated to the different normal forms (reduced systems). The technique is applied to the reduction of a Hamiltonian system modelling the trapping mechanisms for the electron of a strongly ionized hydrogen atom. © 2002

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1. INTRODUCTION

This paper deals with Hamiltonian systems formed by a dominant part, \mathcal{H}_0 , plus a small time-dependent perturbation \mathcal{P} , that is,

$$\mathcal{H}(\mathbf{z}, \mathbf{Z}, t; \varepsilon) = \mathcal{H}_0(\mathbf{z}, \mathbf{Z}) + \mathcal{P}(\mathbf{z}, \mathbf{Z}, t; \varepsilon), \quad (1)$$

where \mathcal{H}_0 and \mathcal{P} are analytic functions in their variables, and coordinates \mathbf{z} and their corresponding moments \mathbf{Z} are vectors in \mathbf{R}^3 , whereas ε represents

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a small dimensionless parameter. More specifically, we consider regular and small perturbations of the Kepler problem; thus,

$$\mathcal{H}_0(\mathbf{z}, \mathbf{Z}) = \frac{1}{2} \mathbf{Z} \cdot \mathbf{Z} - \frac{\mu}{\|\mathbf{z}\|}, \quad (2)$$

where $\|\cdot\|$ denotes the Euclidean norm on \mathbf{R}^3 , (\cdot) the usual scalar product for vectors on \mathbf{R}^3 , and $\mu > 0$ is the gravitational constant with physical dimension $[\text{length}^3/\text{time}^2]$. Besides, $|\mathcal{P}| \ll |\mathcal{H}_0|$. Eq. (2) describes the attraction with respect to a main body (the primary) of a particle of unit mass with position \mathbf{z} and velocity \mathbf{Z} .

Suppose that the primary rotates with a uniform angular speed Ω with physical dimension $[1/\text{time}]$. Let $\{\mathbf{b}_1, \mathbf{b}_2, \mathbf{b}_3\}$ be a three-dimensional reference frame attached to the main body such that \mathbf{b}_3 is the axis of rotation. Furthermore, suppose that the perturbation \mathcal{P} is expressed in the form

$$\mathcal{P}(\mathbf{z}, \mathbf{Z}, t; \varepsilon) \equiv \mathcal{P}[\mathcal{R}(\Omega t, \mathbf{b}_3)(\mathbf{z}), \mathcal{R}(\Omega t, \mathbf{b}_3)(\mathbf{Z}); \varepsilon],$$

where $\mathcal{R}(\psi, \mathbf{y})$ represents the rotation operator about the axis \mathbf{y} with an amplitude ψ . The symplectic map $\sigma: (\mathbf{x}, \mathbf{X}) \rightarrow (\mathbf{z}, \mathbf{Z})$ defined through $\mathbf{x} = \mathcal{R}(\Omega t, \mathbf{b}_3)(\mathbf{z})$ and $\mathbf{X} = \mathcal{R}(\Omega t, \mathbf{b}_3)(\mathbf{Z})$ is used to introduce the reference frame (synodic frame) by means of the vectors $\{\mathbf{s}_1, \mathbf{s}_2, \mathbf{s}_3\}$ as $\mathbf{s}_1 = \mathcal{R}(\Omega t, \mathbf{b}_3)(\mathbf{b}_1)$, $\mathbf{s}_2 = \mathcal{R}(\Omega t, \mathbf{b}_3)(\mathbf{b}_2)$, and $\mathbf{s}_3 = \mathbf{b}_3$. By virtue of σ the perturbation is now independent of the time. Explicitly, one has $\mathcal{P}(\mathbf{z}, \mathbf{Z}, t; \varepsilon) \equiv \mathcal{P}(\mathbf{x}, \mathbf{X}; \varepsilon)$. Moreover, system (1) in the synodic frame is defined by the Hamilton function $\mathcal{H} = \mathcal{H}_0 + \mathcal{P}$:

$$\mathcal{H}(\mathbf{x}, \mathbf{X}; \varepsilon) = \frac{1}{2} \mathbf{X} \cdot \mathbf{X} - \frac{\mu}{\|\mathbf{x}\|} - \Omega (\mathbf{x} \times \mathbf{X}) \cdot \mathbf{s}_3 + \mathcal{P}(\mathbf{x}, \mathbf{X}; \varepsilon). \quad (3)$$

At this point we notice that the passage from the inertial to the synodic frame allows us to make the system defined by \mathcal{H} autonomous by adding the Coriolis term (also called rotating term) $-\Omega (\mathbf{x} \times \mathbf{X}) \cdot \mathbf{s}_3$ to the main part of the Hamiltonian. As we shall see the appearance of the Coriolis term will play the central role when dealing with the normal forms calculations.

Systems of type (3) appear rather frequently in celestial and classical mechanics. We mention the cases of lunar theory [20] and planetary theory [6], the attitude of a nonspherical body rotating in a central field [23], the effect of radiation pressure on the aggregation of dust particles around a rotating planet [18, 21], and artificial satellite theories when the gravitational field of the planet depends on the longitude [11, 16, 37, 40] in celestial mechanics. In classical mechanics we quote the examples of motion of electrons under the influence of electrical or magnetic fields [25] and the generalized rotating van der Waals potential [22] which enlarges some

known polynomial potentials in physical chemistry. All these problems are prototypes of systems which are modelled as perturbations of the two-body Hamiltonian.

Besides, some restricted and full three-body problems are connected to perturbed Kepler systems (see, for instance the pioneering work by Moser [35] and more recent Refs. [26, 27, 32, 34]). In this latter example (three-body problem), since the perturbation is not time-dependent, the rotating term does not appear as a consequence of changing from a fixed to a synodic frame. Indeed, the presence of the Coriolis term is due to the fact that the coordinate system is not Newtonian, but a rotating coordinate system. We refer to the excellent book by Meyer and Hall [34] for thorough studies about three-body problems.

Perturbation theories based on the analysis of the normal forms are commonly used with two purposes:

(a) By the dynamicists with the goal of extracting qualitative information (periodic orbits, invariant tori, stability behaviour, etc.) of the departure Hamiltonian (3); then the normal form is truncated at the order it becomes structurally stable, see [1], and it usually depends on the values taken by the parameters the system depend on; see an example in [46].

(b) By the astronomers, physicists, etc. with the goal of constructing accurate (asymptotic) solutions of system (3); then the normal form is truncated at the order such that the error obtained is small enough. See for instance the papers about the computation of the lunar motion [19, 20].

Thus, it turns out necessary to use a powerful tool capable of computing high-order normal forms of perturbed Keplerian systems for the most general class of Hamiltonians defined by (1).

Typically the use of regularization techniques [28] allows the conversion of the Hamiltonian in a four-dimensional harmonic oscillator (two-dimensional harmonic oscillator if the departure Hamiltonian is also in two dimensions). In this situation the Coriolis term is usually small compared with the Kepler part of \mathcal{H}_0 . Thence, the part coming from the rotation must be placed at first order so that the unperturbed part becomes a four-dimensional harmonic oscillator in 1-1-1-1 resonance (respectively two-dimensional harmonic oscillator in 1-1 resonance) whereas the coupled terms appear at higher orders. Besides, the resulting perturbation is a polynomial in the new variables (four coordinates and four moments plus a constraint among them or two coordinates and two moments). Thus, the calculation of the corresponding normal forms is executed with the usual procedures for normalising Hamiltonian systems in Cartesian or complex-symplectic coordinates (the so-called Birkhoff normalisation [4]). The normal form theorem [34] is the adequate framework where the normalisation is applied. Cases of normal forms of perturbed Keplerian

systems by using Levi-Civita [29] or Kustaanheimo and Stiefel regularizations appear in [26, 27].

However, when the rotating component is not much smaller than the two-body terms, the Kustaanheimo and Stiefel transformation cannot be used to normalise the initial system. This is why we cannot scale the Hamiltonian in such a way that the two-body part be placed at zeroth order and the Coriolis term at first. In this situation an appropriate scaling of our Hamiltonian should take into account the relative sizes of the Coriolis term and the two-body part. Indeed, as Meyer [32] and Meyer and Hall [34] pointed out for some cases of the three-body problem, different scalings (into Poincaré, Hill, and comet's orbits) of the original Hamiltonian function give rise to different types of dynamics. Therefore, different treatments must be accomplished according to the subcases one deals with.

About the relative size of the two parts of the dominant Hamiltonian, three possibilities are in order. Specifically, defining \mathcal{H}_K and \mathcal{H}_C as

$$\mathcal{H}_K(\mathbf{x}, \mathbf{X}) = \frac{1}{2} \mathbf{X} \cdot \mathbf{X} - \frac{\mu}{\|\mathbf{x}\|} \quad \text{and} \quad \mathcal{H}_C(\mathbf{x}, \mathbf{X}) = -\Omega(\mathbf{x} \times \mathbf{X}) \cdot \mathbf{s}_3,$$

and rewriting $\mathcal{H} = \mathcal{H}_0 + \mathcal{H}_1$, one arrives at one of the following scalings:

- (i) $|\mathcal{H}_K| \approx |\mathcal{H}_C|$: both effects are comparable (moderate rotating Kepler problems) and then $\mathcal{H}_0 = \mathcal{H}_K + \mathcal{H}_C$ and $\mathcal{H}_1 = \mathcal{P}$;
- (ii) $|\mathcal{H}_K| \gg |\mathcal{H}_C|$: slow rotations. Then $\mathcal{H}_0 = \mathcal{H}_K$ and $\mathcal{H}_1 = \mathcal{H}_C + \mathcal{P}$;
- (iii) $|\mathcal{H}_K| \ll |\mathcal{H}_C|$: fast rotations. Then $\mathcal{H}_0 = \mathcal{H}_C$ and $\mathcal{H}_1 = \mathcal{H}_K + \mathcal{P}$.

Case (i) is the most difficult one and has not been analysed yet whereas (iii) is the easiest one. In cases (ii) and (iii) other possibilities for \mathcal{H}_1 can be considered according to the relative values of \mathcal{H}_C and \mathcal{P} in (ii) and \mathcal{H}_K and \mathcal{P} in (iii). For instance for situation (ii) one could place \mathcal{H}_C at first order and \mathcal{P} at second (or even higher) order if $|\mathcal{H}_C| \gg |\mathcal{P}|$. However, although each particular problem may present a different scaling at first order and beyond it does not affect the scaling of the zeroth order and we can focus on the three basic choices (i), (ii), and (iii).

Normalisation procedures of perturbed Keplerian systems have been widely used in celestial mechanics. In this context the resort of regularizing and linearizing is sometimes substituted by the use of adequate collections of symplectic variables quite common in astrodynamics. Examples of these correspond to the application of normalisations with special sets of variables, the so-called polar-nodal and Delaunay variables in two and three dimensions [16, 17, 36]. The nonrotating case has been treated in the literature though the extension to systems with rotating unperturbed part is not an easy task. The crucial point is the difficulty arising in the solution of the homology equation, as we shall see in Section 4.

The usual technique to avoid that drawback consists in computing some Fourier and Taylor expansions in some variables (see the classic reference by Tisserand [43]), truncate at an adequate order, and solve the approximate homology equation [5]. This procedure gives satisfactory results when the series expansions are rapidly convergent or when the accuracy needed is not high. However it is more the exception than the rule. Usually, the developments must be carried out with a large amount of terms which, in many cases, leads to obtaining huge and unwieldy formulæ.

In this paper we circumvent the problems described above. The central idea we want to expose is the definition of normal forms by extending two integrals of the unperturbed part (not necessarily the entire unperturbed Hamiltonian) to the normal form system. This method has been proposed in [41] and allows the number of degrees of freedom of the initial system to be reduced by one or two. To carry out this we need to handle polar-nodal and Delaunay variables so as to solve the homology equation in closed form for all coordinates and moments. We shall distinguish three types of unperturbed Hamiltonians according to the three different scalings mentioned before. By doing so the homology equation is solved straightforwardly whether or not the Coriolis part is comparable with the two-body part (i.e., it is either much bigger or much smaller). For the remaining case, the homology equation can still be solved in terms of special functions which extend the generalized incomplete gamma function in \mathbb{C} .

The scaling of \mathcal{H} leads to three types of normal forms. Calculating a normal form implies the introduction of a formal integral in the transformed Hamiltonian. It means that the new Hamiltonian defines a dynamical system of one degree of freedom less than the one that it comes from. Hence, the initial Hamiltonian is reduced by normalising it.

The normal forms we shall compute are not local, but they are defined for the whole phase space where system (1) is defined; i.e., $\mathbf{R}^6 \setminus (\{\mathbf{0}\} \times \mathbf{R}^3)$. Thus, associated to each normalisation we shall construct a four-dimensional phase space. These portraits are parameterized by the sets of invariant functions related to each reduction. For theoretical aspects on the concept of reduction the reader is referred to [30, 31] where the first results on regular reduction (no singular point is present in the reduced phase space) were presented. The book by Abraham and Marsden [1] contains a general review of this regular case. See also Ref. [10] and the examples therein. The joined treatment of regular and singular reductions appears in [3] with the name universal reduction.

Our approach differs in several ways from the standard treatment in the literature:

(a) We scale the initial system to distinguish the three relevant cases. It allows us to isolate the most difficult case and also to treat the three different situations in a sort of unified way.

(b) We make the calculations of the normal forms and generating functions in closed form, avoiding the use of poor convergent series in the eccentricity or the mean anomaly and also huge collections of formulæ. This permits us to deal with very eccentric trajectories (which is actually the case of some artificial satellites, asteroids, etc.).

(c) We do not split a space of functions (where the initial Hamiltonian is defined) as the sum of the kernel and image of a certain linear operator. We shall compute normal forms by extending some integrals of the zeroth order to the whole normal form. In this way we give a method which enlarges the classic approach, providing, therefore, new types of normal forms.

(d) For case (ii) of the above scaling we give an alternative procedure to the usual techniques based on Delaunay normalisation and Kustaanheimo and Stiefel regularizations, giving rise to a systematic and quick way of calculating normal forms and generating functions.

(e) We use invariant theory to define the different reduced phase spaces and normal forms properly. With this we express the normal form completely in terms of the invariants and make a global study of the dynamics associated to each normal form.

The paper has seven sections. Delaunay and polar-nodal variables are presented in Section 2. In Section 3 we explain how to obtain normal forms by extending an integral of the unperturbed part of the system to the transformed Hamiltonian up to a certain order. Section 4 is devoted to the determination of the normal form in closed form when the Coriolis and the two-body components have similar sizes. The solution of the homology equation and the construction of the generating function in terms of generalized incomplete gamma functions in the complex plane are described with detail. Some properties of these functions and recursion formulæ are given as well. Section 5 deals with the calculation of normal forms when the rotations are either very slow or very rapid. The different reduced phase spaces and their corresponding invariants are given in Section 6, where the possibility of a second reduction is discussed. In Section 7 we apply the theory to a hydrogen-like atom in orthogonal electric and magnetic fields.

2. POLAR-NODAL AND DELAUNAY VARIABLES

Polar-nodal variables were introduced by Jacobi, but were used explicitly much later by Whittaker [44], who pointed out their symplectic character.

Let us make first $(x, y, z) = \mathbf{x}$ and $(X, Y, Z) = \mathbf{X}$. The set of orbital coordinates is given by the six-tuple: $(r, \vartheta, \nu, R, \Theta, N)$ where r stands for the radial distance from the origin of reference to the particle, ϑ represents the argument of latitude, and ν is the right ascension of the node whereas R , Θ and N are the conjugate momenta of r , ϑ , and ν respectively. Besides $rR = \mathbf{x} \cdot \mathbf{X}$, the action Θ represents the modulus of the angular momentum vector, i.e., $\Theta = \|\mathbf{x} \times \mathbf{X}\|$ and $N = xY - yX$ stands for the third component of the angular momentum; see more details in [11]. The explicit relation between polar-nodal and Cartesian coordinates is obtained through the following transformation: $q: (r, \vartheta, \nu, R, \Theta, N) \rightarrow (x, y, z, X, Y, Z)$, where

$$\begin{aligned} x &= x' \cos \nu - y' \cos I \sin \nu, & X &= X' \cos \nu - Y' \cos I \sin \nu, \\ y &= x' \sin \nu + y' \cos I \cos \nu, & Y &= X' \sin \nu + Y' \cos I \cos \nu, \\ z &= y' \sin I, & Z &= Y' \sin I, \end{aligned} \quad (4)$$

with $\cos I = N/\Theta$ and x' , y' , X' , and Y' are given by

$$\begin{aligned} x' &= r \cos \vartheta, & X' &= R \cos \vartheta - \frac{\Theta}{r} \sin \vartheta, \\ y' &= r \sin \vartheta, & Y' &= R \sin \vartheta + \frac{\Theta}{r} \cos \vartheta. \end{aligned} \quad (5)$$

We have to take into account that the transformation q is singular for $r = 0$, $\Theta = 0$, and $\Theta = |N|$ as I is an angle defined on $(0, \pi)$. Therefore, the domain of validity of the change given by q is a subset of \mathbf{R}^6 :

$$\Delta_{pn} = (0, +\infty) \times [0, 2\pi) \times [0, 2\pi) \times \mathbf{R} \times (0, \infty) \times (-\Theta, \Theta).$$

Whittaker [44] demonstrated that q is symplectic in Δ_{pn} . Another proof appears in [37]. Indeed, the name of polar-nodal variables is due to the fact that they are constructed as the composition of transformations (4) and (5). Polar-nodal variables are also called Hill or Whittaker variables.

From the above it is readily deduced that polar-nodal variables are not useful for collision ($r \equiv 0$), rectilinear ($\Theta \equiv 0$), and equatorial ($\Theta \equiv |N|$) trajectories. (We call equatorial the orbits satisfying $z = Z = 0$ excluding other cases of $G = |N| = 0$ incorporated into the rectilinear class.) Collision orbits can be studied if the Hamiltonian \mathcal{H} is previously regularized. Rectilinear and equatorial trajectories can still be considered in the normal form context if one resorts to the generators of the reduced phase space, that is, the invariants associated to the reductions. However, polar-nodal and Delaunay variables are better suited than invariants to deal with the normalisation. So, we will pass to them after calculating the normal forms.

The unperturbed part of (3) is now written as $\mathcal{H}_0 = \mathcal{H}_K + \mathcal{H}_C$ with

$$\mathcal{H}_K(r, R, \Theta) = \frac{1}{2} \left(R^2 + \frac{\Theta^2}{r^2} \right) - \frac{\mu}{r} \quad \text{and} \quad \mathcal{H}_C(N) = -\Omega N.$$

Delaunay variables (ℓ, g, h, L, G, H) are a set of action-angle variables defined for Keplerian problems in an elliptic domain (bounded orbits), that is, only for negative values of the energy; see for instance Ref. [1]. They are defined through the polar-nodal variables by means of a generating function. We refer to [16, 17] for more details.

The action L is related to the two-body energy by $-\mu^2/(2L^2) = \mathcal{H}_K$. The moment G designates the modulus of the angular momentum, thus $G \equiv \Theta$. Moment H is the third component of $\mathbf{G} = \mathbf{x} \times \mathbf{X}$, so $H \equiv N$. The coordinate ℓ is an angle called the main anomaly. It is related to the eccentric anomaly E by means of the Kepler equation $\ell = E - e \sin E$, e being the eccentricity of the orbit, which is given by $e = (1 - G^2/L^2)^{1/2}$ and must belong to the interval $[0, 1)$. The angle E is expressed in terms of the f through $\tan(E/2) = ((1-e)/(1+e))^{1/2} \arctan(f/2)$ and the true anomaly is expressed in terms of the radius r by

$$r(1 + e \cos f) = \frac{G^2}{\mu}. \quad (6)$$

Combining adequately the above equations, r is related (implicitly) to ℓ . Also, r and the eccentric anomaly E are connected through the identity

$$r = a(1 - e \cos E), \quad (7)$$

where a represents the semimajor axis of the ellipse. Note that a is related with the moment L by $L^2 = \mu a$. In addition the radial velocity R is also written in terms of the true and eccentric anomalies by means of:

$$R = \frac{\mu e}{G} \sin f \quad \text{and} \quad Rr = Le \sin E. \quad (8)$$

To see how these relations are deduced the reader can consult the books by Smart [42] or Brouwer and Clemence [5].

Angle g is the argument of the pericentre. It is reckoned from the pericentre of the orbit in the instantaneous orbital plane (the one spanned by \mathbf{x} and \mathbf{X}); thus $g = \vartheta - f$. Angle h is the argument of the node; i.e., $h \equiv \nu$.

We pay attention to the limit value $G \equiv L$. For it, $e = 0$ and the argument of the pericentre vanishes. Thus, Delaunay variables are not valid for circular orbits. Besides, we do not consider the trajectories discarded in

Whittaker variables. With this in mind, the domain of validity of Delaunay variables for perturbed Kepler systems \mathcal{H} is given by a subset of \mathbf{R}^6 ,

$$\Delta_D = [0, 2\pi) \times [0, 2\pi) \times [0, 2\pi) \times (0, +\infty) \times (0, L) \times (-G, G).$$

Similar to what we have commented on about polar-nodal variables, the invariants associated to the reductions we shall do will cover the limit cases of rectilinear, equatorial, and circular trajectories. So, we will use first polar-nodal and Delaunay variables to calculate the reducing Hamiltonians and then we will express the reduced flows in the appropriate invariants.

Finally, the zeroth order of (3) is written in Delaunay variables as

$$\mathcal{H}_0(L, H) = \mathcal{H}_K + \mathcal{H}_C = -\frac{\mu^2}{2L^2} - \Omega H.$$

3. LIE TRANSFORMATIONS AND NORMAL FORMS

In a perturbation theory, it is customary to transform an initial Hamiltonian \mathcal{H} into another Hamiltonian \mathcal{K} , the so-called normal form of \mathcal{H} , with the aid of a generating function \mathcal{W} . More specifically the method of Lie transformations [15] can be stated as follows.

Let $\mathbf{x} = (x_1, x_2, \dots, x_{2n})$ and $\mathbf{y} = (y_1, y_2, \dots, y_{2n})$ be two $2n$ -dimensional vectors defined over an open domain of \mathbf{R}^{2n} and such the n first components of \mathbf{x} and \mathbf{y} stand for the coordinates whereas the other n are the associated moments.

An analytic Hamiltonian function depending on a small parameter ε , $\mathcal{H}(\mathbf{x}; \varepsilon) = \sum_{i=0}^{\infty} \varepsilon^i / i! \mathcal{H}_i(\mathbf{x}) \equiv \sum_{i=0}^{\infty} \varepsilon^i / i! \mathcal{H}_i^{(0)}(\mathbf{x})$, is transformed into another Hamiltonian $\mathcal{K}(\mathbf{y}; \varepsilon) = \sum_{i=0}^{\infty} \varepsilon^i / i! \mathcal{K}_i(\mathbf{y}) \equiv \sum_{i=0}^{\infty} \varepsilon^i / i! \mathcal{H}_0^{(i)}(\mathbf{y})$, through a generating function $\mathcal{W}(\mathbf{x}; \varepsilon) = \sum_{i=0}^{\infty} \varepsilon^i / i! \mathcal{W}_{i+1}(\mathbf{x})$, following the recursive formula

$$\mathcal{H}_i^{(j)} = \mathcal{H}_{i+1}^{(j-1)} + \sum_{k=0}^i \binom{i}{k} \{ \mathcal{H}_{i-k}^{(j-1)}, \mathcal{W}_{k+1} \}, \quad (9)$$

with $i \geq 0$, $j \geq 1$. Besides $\{\cdot, \cdot\}$ denotes the Poisson bracket of two functions; see for instance [1]. Hence, Eq. (9) yields the partial differential identity

$$\mathcal{L}_{\mathcal{H}_0}(\mathcal{W}_i) + \mathcal{K}_i = \tilde{\mathcal{H}}_i, \quad (10)$$

where $\tilde{\mathcal{H}}_i$ collects all the terms from the previous order. In this identity, called the homology equation, \mathcal{W}_i and \mathcal{K}_i must be determined according to the specific requirements of the Lie transformation one performs. Symbol $\mathcal{L}_{\mathcal{H}_0}$ designates the Lie operator related to \mathcal{H}_0 ; i.e., $\mathcal{L}_{\mathcal{H}_0}(P) = \{P, \mathcal{H}_0\}$.

The transformation $\mathbf{x} = \mathbf{X}(\mathbf{y}; \varepsilon)$ which relates the old variables, \mathbf{x} , with the new ones, \mathbf{y} , is a near-identity symplectic change of variables. Explicitly, the direct change is given by

$$\mathbf{x} = \mathbf{y} + \sum_{i=1}^{\infty} \frac{\varepsilon^i}{i!} \mathcal{L}_{\mathcal{W}}^i(\mathbf{y}), \quad (11)$$

where the Lie operator applied to a vector \mathbf{y} means that it is applied to each component of it. Besides, the notation $\mathcal{L}_{\mathcal{W}}^i$ refers to the application of $\mathcal{L}_{\mathcal{W}}$ i times. Consequently, Eq. (11) gives the set of variables \mathbf{x} in terms of \mathbf{y} with the use of the generating function \mathcal{W} . Realize that Eq. (11) must be used to transform any function expressed in the old variables \mathbf{x} as a function of the new variables \mathbf{y} .

The inverse transformation, $\mathbf{y} = \mathbf{Y}(\mathbf{x}; \varepsilon)$, is defined as

$$\mathbf{y} = \mathbf{x} + \sum_{i=1}^{\infty} \frac{\varepsilon^i}{i!} \mathcal{L}_{-\mathcal{W}}^i(\mathbf{x}), \quad (12)$$

where $\mathcal{L}_{-\mathcal{W}}$ refers to the Lie operator $\mathcal{L}_{-\mathcal{W}}: F \rightarrow \{\mathcal{W}, F\}$ whereas $\mathcal{L}_{-\mathcal{W}}^i$ refers to the application of $\mathcal{L}_{-\mathcal{W}}$ i times.

Note that Eq. (11) can be used to transform any function expressed in the old variables \mathbf{x} as a function of the new variables \mathbf{y} . Similarly, Eq. (12) is used to transform any function in \mathbf{y} as a function of \mathbf{x} . For this we need to know explicitly the generating function \mathcal{W} . Furthermore, if the solution of a certain normal form \mathcal{K} (as an ordinary differential equation, e.g., an explicit expression of the vector $\mathbf{y}(t)$) were known and we were interested in obtaining a formal and explicit solution of the departure Hamiltonian \mathcal{H} , e.g., an explicit expression of the vector $\mathbf{x}(t)$, we should make use of (12), as is typically done in astrodynamics.

The above method is formal as the convergence of the various series is not discussed. Moreover, the series usually diverge. However, the first orders of the transformed system give interesting information and the process will be stopped at a certain order M . Thus, these terms of the series are useful to construct the transformed Hamiltonian and the generating function since they are unaffected by the divergent character of the whole process. See [33] for a very elegant treatment of Lie transformations.

We use Lie transformations to build normal forms. Specifically, our programme consists in extending an integral of the dominant Hamiltonian to the normal form. We present it from a perspective of formal results, again following Meyer.

THEOREM 3.1. *Let $M \geq 1$ be given, let $\{\mathcal{A}_i\}_{i=0}^M$, $\{\mathcal{B}_i\}_{i=1}^M$ and $\{\mathcal{C}_i\}_{i=1}^M$ be sequences of linear spaces of smooth functions defined on a common domain*

Δ in \mathbf{R}^{2n} , and let \mathcal{J} be a function in \mathcal{A}_j , for some $j \geq 0$, with the following properties:

- (a) $\mathcal{B}_i \subseteq \mathcal{A}_i$, $i = 1, \dots, M$;
- (b) $\mathcal{H}_i \in \mathcal{A}_i$, $i = 0, 1, \dots, M$;
- (c) $\{\mathcal{A}_i, \mathcal{C}_j\} \subseteq \mathcal{A}_{i+j}$, $i+j = 1, \dots, M$;
- (d) $\forall D \in \mathcal{A}_i$, $i = 1, \dots, M$, one can find $E \in \mathcal{B}_i$ and $F \in \mathcal{C}_i$ such that

$$E = D + \{\mathcal{H}_0, F\} \quad \text{and} \quad \{\mathcal{J}, E\} = 0.$$

Then, there exists an analytic function \mathcal{W} ,

$$\mathcal{W}(\mathbf{x}; \varepsilon) = \sum_{i=0}^{M-1} \frac{\varepsilon^i}{i!} \mathcal{W}_{i+1}(\mathbf{x}),$$

with $\mathcal{W}_i \in \mathcal{C}_i$, $i = 1, \dots, M$, such that the change of variables $\mathbf{x} = \mathbf{X}(\mathbf{y}; \varepsilon)$ is a formal symplectic \mathcal{C}^M -diffeomorphism. Besides this change is the general solution of the I.V.P. $d\mathbf{x}/d\varepsilon = \mathcal{J} \partial \mathcal{W}(\mathbf{x}; \varepsilon) / \partial \mathbf{x}$, $\mathbf{x}(0) = \mathbf{y}$ (\mathcal{J} being the skew-symmetric matrix of dimension $2n$) and transforms Hamiltonian

$$\mathcal{H}(\mathbf{x}; \varepsilon) = \sum_{i=0}^{\infty} \frac{\varepsilon^i}{i!} \mathcal{H}_i(\mathbf{x})$$

to the convergent Hamiltonian (the normal form)

$$\mathcal{K}(\mathbf{y}; \varepsilon) = \sum_{i=0}^M \frac{\varepsilon^i}{i!} \mathcal{K}_i(\mathbf{y}) + \mathcal{O}(\varepsilon^{M+1}),$$

with $\mathcal{K}_i \in \mathcal{B}_i$ and $\{\mathcal{K}_i, \mathcal{J}\} = 0$, $i = 1, \dots, M$. Besides, if $\{\mathcal{H}_0, \mathcal{J}\} = 0$, then \mathcal{J} is a formal integral of \mathcal{K} .

Proof. It appears in [41] and is based on results reported in [33, 34]. ■

Our Theorem 3.1, a consequence of the result given in Meyer and Hall in [34, Chap. VIII, Corollary 2, p. 217], is that we need an integral of the dominant part \mathcal{J} and, in hypothesis (d) we have added that there must be a function E such that $\{\mathcal{J}, E\} = 0$. This is the key point to extend the integral \mathcal{J} to the truncated normal form \mathcal{K} and, consequently, to apply reduction techniques. In practice, however, many difficulties can arise when trying to solve the homology equation needed to calculate a certain \mathcal{W}_i as we will see later.

Another remark is that whenever \mathcal{J} is an integral of \mathcal{H}_0 , the effect of constructing \mathcal{K} , where $\mathcal{K}_i \in \ker(\mathcal{L}_{\mathcal{J}})$ for $i = 1, \dots, M$, is to extend (formally) the integral of the unperturbed system to the whole transformed Hamiltonian \mathcal{K} . It means that the choice of \mathcal{J} can be done adequately if

one knows previously the integrals of \mathcal{H}_0 . In these cases we obtain an integral of the truncated system \mathcal{H} independent of it, and therefore, the number of degrees of freedom of \mathcal{H} has been reduced by one after the transformation. Moreover, the function $\mathcal{J}^*(\mathbf{x}; \varepsilon) = \mathcal{J}(\mathbf{x}) + \sum_{i=1}^M \varepsilon^i / i! \mathcal{L}_{\mathcal{W}}^i [\mathcal{J}(\mathbf{x})]$ becomes a formal integral of \mathcal{H} (with an approximation of $\mathcal{O}(\varepsilon^{M+1})$) functionally independent of it. Note that $\mathcal{J}(\mathbf{x}) \equiv \mathcal{J}(\mathbf{y})$.

Let us emphasize that \mathcal{C}_i are not necessarily equal to \mathcal{A}_i , though they satisfy hypothesis (c) of Theorem 3.1; thus the calculation of the normal form \mathcal{H} does not imply a decomposition of the spaces \mathcal{A}_i . This fact allows us to consider the generating functions \mathcal{W} in different spaces of functions from those corresponding to \mathcal{H} and \mathcal{H} .

4. NORMAL FORMS FOR MODERATE ROTATING KEPLER PROBLEMS

4.1. *Solution of the Homology Equation*

We apply now the results of Section 3 to our Hamiltonians. In many problems—for instance, those mentioned in the Introduction as prototypes in classical and celestial mechanics—the initial Hamiltonian is expressed in a combination of polar-nodal and Delaunay variables as $\mathcal{H} = \mathcal{H}_0 + \mathcal{H}_1$ where

$$\mathcal{H}_0(L, H) = -\frac{\mu^2}{2L^2} - \Omega H \quad (13)$$

and

$$\mathcal{H}_1(\ell, g, h, L, G, H) = \sum_{\substack{(j, l, m) \in \mathbb{Z}^3 \\ k \in \mathbb{Z}^+ \cup \{0\}}} \left(\frac{r}{\alpha}\right)^j \left(\frac{R}{\beta}\right)^k \mathcal{F}^{(l, m)}(g, h, L, G, H), \quad (14)$$

with

$$\begin{aligned} \mathcal{F}^{(l, m)}(g, h, L, G, H) &= \mathcal{C}^{(l, m)}(L, G, H) \cos(lg + mh) \\ &\quad + \mathcal{S}^{(l, m)}(L, G, H) \sin(lg + mh). \end{aligned}$$

The quantities α and β are nonnull real constants having physical dimensions [length] and [length/time], respectively. Besides k is always a non-negative integer. The functions $\mathcal{C}^{(l, m)}$ and $\mathcal{S}^{(l, m)}$ are analytic in the moments. Realize that r and R are not explicitly written in terms of Delaunay variables but both depend on the variables ℓ , L and G . With the

use of expressions (6), (7), and (8) the angles f and E could appear explicitly in \mathcal{H}_1 . Indeed, the cosines and sines of E and f are related to the powers of r and R through:

$$\cos E = \frac{1}{e} \left(1 - \frac{r}{a} \right), \quad \sin E = \frac{rR}{Le}, \quad \cos f = \frac{1}{e} \left(\frac{G^2}{\mu r} - 1 \right), \quad \sin f = \frac{GR}{\mu e}.$$

Thus, there is no loss of generality by taking formula (14). This is the most general situation we are going to treat along this work. Note that we can define \mathcal{A}_0 and \mathcal{A}_1 as the space of smooth functions in an open domain of \mathbf{R}^6 and consider $\mathcal{H}_0 \in \mathcal{A}_0$ and $\mathcal{H}_1 \in \mathcal{A}_1$. A typical system having (13) as the unperturbed Hamiltonian arises in the spatial restricted three body problem when the infinitesimal particle is not very near one of the primaries, the so-called Poincaré's orbits [34].

As pointed out in Section 1, further scaling of \mathcal{H}_1 could be performed depending on the requirements on each particular problem. However, it would not alter the treatment we will follow through the paper, as we are interested only in the separation between orders zero and one.

Now, for a given $i \geq 1$ the homology equation (10) in this case is

$$\mathcal{L}_{\mathcal{H}_0}(\mathcal{W}_i) + \mathcal{K}_i \equiv n \frac{\partial \mathcal{W}_i}{\partial \ell} - \Omega \frac{\partial \mathcal{W}_i}{\partial h} + \mathcal{K}_i = \tilde{\mathcal{H}}_i, \quad (15)$$

where $n = \mu^2/L^3$ represents the mean motion of the particle orbiting around the main body. Its physical dimension is [1/time].

In (15), $\tilde{\mathcal{H}}_i$ is supposed to be a function like (14) whereas \mathcal{W}_i and \mathcal{K}_i are the unknowns. The way we proceed consists in selecting an adequate \mathcal{K}_i and solving the partial differential equation for \mathcal{W}_i . The difficulty arises when trying to solve Eq. (15) for \mathcal{W}_i . The reason is that the terms $\tilde{\mathcal{H}}_i$ and \mathcal{K}_i do not contain explicit expressions of ℓ . In fact, in most situations the mean anomaly is present only through r and R . Note that \mathcal{W}_i belongs to a space of smooth functions \mathcal{C}_i not necessarily \mathcal{A}_i and $\mathcal{K}_i \in \mathcal{B}_i \subseteq \mathcal{A}_i$.

First we focus on the calculation of \mathcal{K}_i . Since the normal form of \mathcal{H} is a Hamiltonian \mathcal{K} such that it defines a system of two degrees of freedom, the usual way consists in building \mathcal{K} step by step such that $\{\mathcal{H}_0, \mathcal{K}_i\} = 0$ up to a certain $M \geq 1$; it means that \mathcal{H}_0 will become an integral of the truncated normalised system. However, it is possible to obtain other normal forms using different functions \mathcal{J} satisfying $\{\mathcal{H}_0, \mathcal{J}\} = 0$. Indeed, one should construct a Hamiltonian \mathcal{K} such that $\{\mathcal{J}, \mathcal{K}_i\} = 0$ up to $i = M$. By doing so \mathcal{J} would become an integral of \mathcal{K} after truncation. Two appropriate choices are $\mathcal{J} = -\mu^2/(2L^2)$ or $\mathcal{J} = -\Omega H$ but other combinations could be taken as well. Here we take only the three possibilities mentioned. The theoretical concepts about the construction of normal forms for a given integral of the unperturbed Hamiltonian are performed in Ref. [41].

We have to explain how to obtain each part \mathcal{K}_i together with its companion \mathcal{W}_i for the three choices of \mathcal{J} . We start with the calculation of \mathcal{K}_i . Specifically, if we take a nonnull term $P_i^{(j,k,l,m)}$ of $\tilde{\mathcal{H}}_i$ we have to decide whether this term contributes to \mathcal{K}_i —with the term $P_i'^{(j,k,l,m)}$ obtained from $P_i^{(j,k,l,m)}$ —or if its contribution is null. Besides, in both cases we need to calculate its antiimage, i.e., the function $Q_i^{(j,k,l,m)}$ such that $\mathcal{L}_{\mathcal{H}_0}(Q_i^{(j,k,l,m)}) + P_i'^{(j,k,l,m)} = P_i^{(j,k,l,m)}$.

For convenience of notation we first pass (14) to complex variables. Hence, instead of $\mathcal{F}^{(l,m)}$ we use $\mathcal{G}^{(l,m)}$ defined as $\mathcal{G}_i^{(l,m)}(g, h, L, G, H) = \mathcal{E}_i^{(l,m)}(L, G, H) \exp[\imath(lg + mh)]$, (\imath denotes $\sqrt{-1}$). The perturbation is

$$\mathcal{R}_i(\ell, g, h, L, G, H) = \sum_{\substack{(j,l,m) \in \mathbb{Z}^3 \\ k \in \mathbb{Z}^+ \cup \{0\}}} \left(\frac{r}{\alpha}\right)^j \left(\frac{R}{\beta}\right)^k \mathcal{G}_i^{(l,m)}(g, h, L, G, H), \quad (16)$$

where k is a nonnegative integer. Note that $\mathcal{E}_i^{(l,m)}$ depends only on the moments. Now, we write $P_i^{(j,k,l,m)}$ as

$$P_i^{(j,k,l,m)}(\ell, g, h, L, G, H) = \left(\frac{r}{\alpha}\right)^j \left(\frac{R}{\beta}\right)^k \mathcal{G}_i^{(l,m)}(g, h, L, G, H). \quad (17)$$

Now we are ready to perform the normalisation for the three cases of \mathcal{J} .

(i) If $\mathcal{J} \equiv \mathcal{H}_0$ then the condition for a term $P_i^{(j,k,l,m)}$ like (17) to belong to the normal form system is that $\{\mathcal{H}_0, P_i^{(j,k,l,m)}\} = 0$. With the aid of the partial derivatives of r and R with respect to ℓ and h (see cf. [16] or [37]),

$$\frac{\partial r}{\partial \ell} = \frac{L^3 R}{\mu^2}, \quad \frac{\partial r}{\partial h} = 0, \quad \frac{\partial R}{\partial \ell} = \frac{L^3}{\mu r^2} \left(\frac{G^2}{\mu r^2} - 1 \right), \quad \frac{\partial R}{\partial h} = 0,$$

one has that

$$\{\mathcal{H}_0, P_i^{(j,k,l,m)}\} = \left[\frac{k(\mu r - G^2) - jr^2 R^2}{r^3 R} + im\Omega \right] P_i^{(j,k,l,m)}.$$

Thus, $\{\mathcal{H}_0, P_i^{(j,k,l,m)}\} = 0$ if and only if $m\Omega = 0$ and $k(\mu r - G^2) = jr^2 R^2$. Note, however that a linear combination $T_i = \sum_{j,k,l,m} P_i^{(j,k,l,m)}$ could yield that $\{\mathcal{H}_0, T_i\} = 0$. To simplify a little, we can consider terms T_i such that its indexes l and m are fixed. The reason is that $\mathcal{L}_{\mathcal{H}_0}$ is a linear operator with diagonal block for the angles ℓ , g , and h but not for r and R . It means that given a term $U(L, G, H) \exp[\imath(j_1 \ell + j_2 g + j_3 h)]$, its corresponding image by $\mathcal{L}_{\mathcal{H}_0}$ is the term $\imath(j_1 n - j_3 \Omega) U(L, G, H) \exp[\imath(j_1 \ell + j_2 g + j_3 h)]$. Thus, for a combination of terms $P_i^{(j,k,l,m)}$ we only need to consider the variation of the

indexes $j \in \mathbf{Z}$ and $k \in \mathbf{Z}^+ \cup \{0\}$, that is, those indexes affecting the variables r and R . So, we take $0 \neq T_i^{(l,m)} = \sum_{j,k} P_i^{(j,k,l,m)}$, obtaining

$$\{\mathcal{H}_0, T_i^{(l,m)}\} = \sum_{j,k} \frac{k(\mu r - G^2) - jr^2 R^2}{r^3 R} P_i^{(j,k,l,m)} + im\Omega T_i^{(l,m)}.$$

Thus, the conditions for the existence of (exact) resonances are

$$\sum_{j,k} [k(\mu r - G^2) - jr^2 R^2] P_i^{(j,k,l,m)} = 0 \quad \text{and} \quad m\Omega = 0, \quad (18)$$

provided that $r \neq 0$ and $R \neq 0$. Hence, we notice that for the nonrotating case, a term $T_i^{(l,m)}$ to belong to \mathcal{H}_i must satisfy $m \equiv 0$. That is, $T_i^{(l,m)}$ must be independent of h . It implies that with this choice of \mathcal{I} one obtains in a unique process two integrals: (1) \mathcal{H}_0 (or L) and (2) $-\Omega H$ (or H). Consequently, this case becomes a particular situation of (ii) and (iii). This fact turns out to be rather surprising and had not been clarified up to now. Moreover, as the way of calculating $P_i^{(l,m)}$ would be complicated we discard this option. So, we leave the terms $T_i^{(l,m)}$ going back to $P_i^{(j,k,l,m)}$.

(ii) If $\mathcal{I} \equiv -\mu^2/(2L^2)$, then a term $P_i^{(j,k,l,m)}$ belongs to \mathcal{H} if and only if the partial derivative $\partial P_i^{(j,k,l,m)}/\partial \ell = 0$. The way of calculating the corresponding $P_i^{(j,k,l,m)}$ is by computing the following average with respect to the mean anomaly, $P_i^{(j,k,l,m)} = \frac{1}{2\pi} \int_0^{2\pi} P_i^{(j,k,l,m)} d\ell$. The process is not immediate and we shall discuss it a bit in Section 4.1, since there we shall calculate $P_i^{(j,k,l,m)}$ in the same way as here. The reader is also referred to cf. [5, 17, 36]. Note that as $P_i^{(j,k,l,m)}$ is independent of ℓ , the normal form \mathcal{H} will have L as a new integral.

(iii) If $\mathcal{I} \equiv -\Omega H$, then a term $P_i^{(j,k,l,m)}$ belongs to \mathcal{H} if and only if $\partial P_i^{(j,k,l,m)}/\partial h = 0$. This time, $P_i^{(j,k,l,m)}$ is simply calculated by computing the average with respect to the argument of the node, i.e., $P_i^{(j,k,l,m)} = \frac{1}{2\pi} \int_0^{2\pi} P_i^{(j,k,l,m)} dh$. Hence, as $P_i^{(j,k,l,m)}$ is independent of h , H will become an integral of the normal form \mathcal{H} . Note that \mathcal{H} will be axially symmetric with respect to the axis z .

The manner of calculating the antiimage of $P_i^{(j,k,l,m)}$ for cases (ii) and (iii) must be indicated. Let \mathcal{R}_i be given by $\mathcal{R}_i = \tilde{\mathcal{H}}_i - \mathcal{H}_i$ where \mathcal{H}_i has been calculated via an average either over ℓ or over h . We need to obtain \mathcal{W}_i which is the antiimage of \mathcal{R}_i with respect to the operator $\mathcal{L}_{\mathcal{H}_0}$. It is equivalent to solve (15) when \mathcal{R}_i corresponds to a function of the type (14).

Now, we want to solve Eq. (15) for a certain function $\tilde{\mathcal{H}}_i$ of the type (16). The following proposition gives the key to obtain an explicit expression of the generating function in terms of some special functions.

PROPOSITION 4.1. *Let $\mathcal{R}_i = \tilde{\mathcal{K}}_i - \mathcal{K}_i$ be a function of the type (16). Then Eq. (15) is satisfied if \mathcal{W}_i is taken as*

$$\mathcal{W}_i(\ell, g, h, L, G, H) = \sum_{\substack{(j, l, m) \in \mathbb{Z}^3 \\ k \in \mathbb{Z}^+ \cup \{0\}}} \left\{ \mathcal{T}_i^{(j, k, l, m)}(\ell, g, h, L, G, H) \times \int_0^\ell [r(t)]^j [R(t)]^k \exp(-im\Omega t/n) dt \right\}, \quad (19)$$

where

$$\mathcal{T}_i^{(j, k, l, m)}(\ell, g, h, L, G, H) = \frac{\mathcal{G}_i^{(l, m)}(L, G, H)}{n\alpha^j \beta^k} \exp[\iota(lg + mh + m\Omega\ell/n)].$$

Proof. As the partial differential equation (15) is linear, one needs to calculate the Charpit equations, transforming the determination of a complete integral of (15) into the calculation of a primitive. The solution has been found with the aid of the built-in function `DSolve` of Mathematica, Version 4.0. See Ref. [45] for a whole description of the function `DSolve`. ■

Note that when $j = k = 0$ but $m \neq 0$ and $\Omega \neq 0$, \mathcal{W}_i is easily obtained by computing the corresponding quadrature with respect to t . However, when $jk \neq 0$, and $m\Omega \neq 0$ we have to deal with cumbersome integrals.

Let us observe that it could be possible to generalize the above result in the sense that the Coriolis term $-\Omega H$ could be substituted by any function $F(G, H)$. Thus, Proposition 4.1 can be adapted to treat systems whose dominant part is given by $\mathcal{H}_0(L, G, H) = -\mu^2/(2L^2) + F(G, H)$. The resulting formulæ are more involved than in the case $F(G, H) = -\Omega H$ but the type of integrals which should be analysed is exactly the same as in Eq. (19). A particular situation refers to the case of perturbation of planar Keplerian problems with rotation. Then, employing the planar version of the Delaunay variables, that is, the set defined by ℓ , g , L , and G , the dominant part of a certain Hamiltonian is $F(G, H) = -\Omega G$. (See the book by Meyer and Hall [34] for details.) Then, the latter proposition applies after substituting h and H by g and G , respectively. Besides, in this two-dimensional case, the variables h and H are simply dropped.

4.2. Change of Variables

The implicit solution of Eq. (15) given by Proposition 4.1 can be slightly improved by introducing special functions. The idea consists in performing an adequate change of variables in order to simplify the integral expression appearing in (19). Indeed, if we define the complex variable z_E as $z_E = \exp(\iota E)$, we have to substitute the functions related to the mean anomaly

by their equivalent expressions in terms of z_E . This change was first proposed by Hansen with the idea of performing double expansions of some functions depending on ℓ (Fourier developments in the mean anomaly together with Taylor developments in the eccentricity); with regard to this point, see e.g., Smart [42, p. 31]. Further, they are used to integrate in closed form the equation of the centre ($\varphi = f - \ell$) and other related functions with the dilogarithm [36] and the generalization to the polylogarithm of any order [38] and [39] in perturbed Keplerian systems with nonrotating primary body. We shall go back to this point as a particular situation of the more general case treated in this paper.

We define first an auxiliary variable; specifically, the eccentricity function $\eta = \sqrt{1 - e^2}$ with $\eta \in (0, 1)$ as we consider only bounded orbits. It depends solely on the moments L and G . Now, the variables related with ℓ are written in terms of E as:

$$\begin{aligned}\ell &= \imath \left[\frac{e}{2} (z_E - z_E^{-1}) - \log(z_E) \right], & E &= -\imath \log(z_E), \\ f &= \imath \log \left[\frac{-ez_E + 1 + \eta}{(1 + \eta) z_E - e} \right], & r &= \frac{L^2}{2\mu} (-ez_E + 2 - ez_E^{-1}), \\ R &= \frac{\imath \mu e (z_E^2 - 1)}{L(ez_E^2 - 2z_E + e)},\end{aligned}\quad (20)$$

Now, \mathcal{W}_i is written in terms of z_E as we state next.

PROPOSITION 4.2. *Let us suppose hypotheses of Proposition 4.1 hold. If $j \neq 0$ or $k \neq 0$ or $j \neq 0$ and $k \neq 0$, the part of the generating function at order i , that is, \mathcal{W}_i , can be written as*

$$\begin{aligned}\mathcal{W}_i(z_E, g, h, L, G, H) &= \sum_{\substack{(j, l, m) \in \mathbb{Z}^3 \\ k \in \mathbb{Z}^+ \cup \{0\}}} \left\{ \mathcal{W}_i^{(j, k, l, m)}(z_E, g, h, L, G, H) \right. \\ &\quad \left. \times \int_1^{z_E} \mathcal{V}_i^{(j, k, m)}(w, L, G) \exp \left[\frac{m\Omega e}{2n} (w - w^{-1}) \right] dw \right\},\end{aligned}\quad (21)$$

where

$$\begin{aligned}\mathcal{W}_i^{(j, k, l, m)}(z_E, g, h, L, G, H) \\ = \frac{(-1)^j \imath^{1+k}}{2^{1+j} \alpha^j \beta^k \mu^{2+j-k}} \frac{e^k \mathcal{G}_i^{(l, m)}(L, G, H)}{L^{-3-2j+k}} \exp \{ \imath [lg + mh + m\Omega \ell(z_E)/n] \}\end{aligned}$$

and

$$\mathcal{V}_i^{(j,k,m)}(w, L, G) = \frac{(w^2 - 1)^k (ew^2 - 2w + e)^{1+j-k}}{w^{2+j+m\Omega/n}}.$$

Moreover the complex integral of (21) must be interpreted as a path integral, the path being a part or the entire unit circle connecting 1 and z_E .

Proof. After performing the change $w = z_E$, one computes the differential dt . From the expression of ℓ in terms of z_E one can calculate the differential $d\ell$, yielding $d\ell = -ir/(a z_E) dz_E$. From here, it is easily deduced that $dt = \frac{i}{2} (ew^2 - 2w + e) w^{-2} dw$. Now, for the limits of integrals, one has that $t = 0$ if and only if $w = 1$ and $t = \ell$ if and only if $w = z_E$. Thus, taking into account (20) and that $L^2 = \mu a$, the expression for \mathcal{W}_i given by (19) is converted into formula (21). ■

The main consequence of Proposition 4.2 is that the solution of the homology equation (15) is given in closed form through the primitives appearing in Eq. (21); in other words, the solution is obtained for all $e \in (0, 1)$ and for all $\ell \in [0, 2\pi)$ in terms of integrals of the type

$$I_0^{(b,m)}(z_E, L, G; \mu, \Omega) = \int_1^{z_E} w^b \exp \left[\frac{m\Omega e}{2n} (w - w^{-1}) \right] dw, \quad (22)$$

when $j+1 \geq k$ in $\mathcal{V}_i^{(j,k,m)}$ and of the form

$$I_1^{(b,m,p)}(z_E, L, G; \mu, \Omega) = \int_1^{z_E} \frac{w^b \exp \left[\frac{m\Omega e}{2n} (w - w^{-1}) \right]}{[(1+\eta)w - e]^p} dw, \quad (23)$$

$$I_2^{(b,m,p)}(z_E, L, G; \mu, \Omega) = \int_1^{z_E} \frac{w^b \exp \left[\frac{m\Omega e}{2n} (w - w^{-1}) \right]}{(-ew + 1 + \eta)^p} dw,$$

when $j+1 < k$ in $\mathcal{V}_i^{(j,k,m)}$. Notice that z_E is a complex variable in the unit circle centred at the origin of the complex plane. We have not found the solution of the latter integrals with the aid of either elementary or special functions. However, we shall define some useful functions in terms of (22) and (23). Thence, we shall express the generating function \mathcal{W} in terms of those functions. By means of the change $u = 1/w$ in (22) and (23), one arrives at:

- (i) $I_0^{(b,m)}(z_E, L, G; \mu, \Omega) = -I_0^{(-b-2, -m)}(z_E^{-1}, L, G; \mu, \Omega);$
- (ii) $I_1^{(b,m,p)}(z_E, L, G; \mu, \Omega) = -I_2^{(-b+p-2, -m, p)}(z_E^{-1}, L, G; \mu, \Omega);$
- (iii) $I_2^{(b,m,p)}(z_E, L, G; \mu, \Omega) = -I_1^{(-b+p-2, -m, p)}(z_E^{-1}, L, G; \mu, \Omega).$

So, we can use only $I_0^{(b,m)}$ and $I_1^{(b,m,p)}$. The relation between the coefficients j and k , that is to say, the inequalities $1+j-k \geq 0$ or $1+j-k < 0$, must be understood as a generalization of the so-called D'Alembert characteristic (see Ref. [5] for a detailed discussion of this). Here we shall say that the pair (j, k) verifies the D'Alembert characteristic (for moderate rotating Kepler problems) when both j and k are integer numbers such that $k \geq 0$ and $1+j-k \geq 0$. In all the cases b is a real number; furthermore, $b = q - m\Omega/n$ with q an integer. The parameter p stands always for a nonnegative integer. Realize that the denominators $(1+\eta)w - e$ and $-ew + 1 + \eta$ appear after decomposing the quadratic polynomial $ew^2 - 2w + e$ in polynomials of degree one in w and relating the functions η and e .

We give now a couple of examples showing how the integrals $I_0^{(b,m)}$, $I_1^{(b,m,p)}$ are introduced when dealing with functions depending on h in the case $\Omega \neq 0$. Taking $\mathcal{R}_i(\ell, h, L, G) = (\alpha^2/r^2) \sin(3h)$ where α stands for a constant with dimension [length] and $\mathcal{I} = -\Omega H$, we have that $\mathcal{R}'_i = \frac{1}{2\pi} \int_0^{2\pi} \mathcal{R}_i dh = 0$ and the generating function is

$$\begin{aligned} \mathcal{W}_i(\ell, h, L, G) &= \frac{\alpha^2}{2\eta L} \{ (1+\eta) [\exp(3i(s\ell+h)) I_1^{(-3s, 3, 1)}(z_E(\ell), L, G; \mu, \Omega) \\ &\quad - \exp(-3i(s\ell+h)) I_1^{(3s, -3, 1)}(z_E(\ell), L, G; \mu, \Omega)] \\ &\quad + e [\exp(-3i(s\ell+h)) I_1^{(-3s-1, 3, 1)}(z_E^{-1}(\ell), L, G; \mu, \Omega) \\ &\quad - \exp(3i(s\ell+h)) I_1^{(3s-1, -3, 1)}(z_E^{-1}(\ell), L, G; \mu, \Omega)] \}, \end{aligned}$$

where s denotes the quotient $L^3\Omega/\mu^2$. Note that in this particular case $j = -2$ and $k = 0$, therefore, only functions of the type $I_1^{(b,m,p)}$ can appear in \mathcal{W}_i . Thus, the functions \mathcal{R}_i and \mathcal{W}_i satisfy the partial differential identity $\mathcal{L}_{\mathcal{H}_0}(\mathcal{W}_i) = \mathcal{R}_i$.

Let \mathcal{I} be $-\mu^2/(2L^2)$ and $\mathcal{R}_i(\ell, h, L, G) = (\alpha/r) \cos(2h)$ with the same physical dimension for α as before. Now, the corresponding generating function reads as

$$\begin{aligned} \mathcal{W}_i(\ell, h, L, G) &= -\frac{\alpha L i}{2\mu} \{ \exp[2i(s\ell+h)] I_0^{(-2s-1, 2)}(z_E(\ell), L, G; \mu, \Omega) \\ &\quad + \exp[-2i(s\ell+h)] I_0^{(2s-1, -2)}(z_E(\ell), L, G; \mu, \Omega) \} \\ &\quad - \frac{\alpha L}{\mu} \cos(2h), \end{aligned}$$

with the same notation for s as before. Besides, the average of $\mathcal{R}_i(\ell, h, L, G)$ with respect to ℓ is $\mathcal{R}'_i(h, L) = \alpha\mu L^{-2} \cos(2h)$. Note that \mathcal{W}_i is

a real function. It can be verified after expanding it as a Fourier series as the complex part of the series vanishes. In addition to that the Fourier approximation also shows that \mathcal{W}_i is periodic with respect to ℓ (or to E). This time as $j = -1$ and $k = 0$ then $j+1 \geq k$ and functions of the type $I_0^{(b,m)}$ in the generating function are expected. The partial differential identity $\mathcal{L}_{\mathcal{H}_0}(\mathcal{W}_i) + \mathcal{R}_i' = \mathcal{R}_i$ is satisfied.

Integrals (22) and (23) admit other expressions after making the change of variable $w = \exp(i\kappa)$:

$$I_0^{*(b,m)}(E, L, G; \mu, \Omega) = i \int_0^E \exp \left\{ i \left[(b+1) \kappa + \frac{m\Omega e}{n} \sin \kappa \right] \right\} d\kappa,$$

$$I_1^{*(b,m,p)}(E, L, G; \mu, \Omega) = i \int_0^E \frac{\exp \left\{ i \left[(b+1) \kappa + \frac{m\Omega e}{n} \sin \kappa \right] \right\}}{[(1+\eta) \exp(i\kappa) - e]^p} d\kappa.$$

We do not know how to solve $I_0^{*(b,m)}$ and $I_1^{*(b,m,p)}$. In fact, the only result we have is that, whether b is an integer, $I_0^{*(b,m)}(2\pi, L, G; \mu, \Omega) = \frac{i}{2\pi} J_{b+1}[(1+b)m\Omega e/n]$. Here $J_n(z)$ denotes Bessel function of the first kind; see [2].

Nevertheless, these integrals can be used to make numerical integrations to approximate $I_0^{(b,m)}$ and $I_1^{(b,m,p)}$ for fixed values of b, m, n, Ω, e , and E .²

For nonrotating problems ($\Omega \equiv 0$), the exponential term of the integrals $I_0^{(b,m)}$ and $I_1^{(b,m,p)}$ vanishes. Besides, the exponent b of w is always an integer. Hence, the resulting integrands are rational functions in the variable w . This feature simplifies considerably the way of obtaining closed-form solutions for \mathcal{W}_i compared to how it was achieved in Ref. [36]. A detailed analysis of the calculation of normal forms for nonrotating Kepler systems in three dimensions following a similar approach to what is presented here appears in [39]. In addition, if the rotation is very slow compared with the Keplerian part the primitives which have to be solved have rational integrals, yielding closed solutions for the functions \mathcal{W}_i .

Another case for which the calculation of integrals (22) and (23) is reduced to calculate primitives of rational functions occurs when the index

² When dealing with the functions $I_0^{(b,m)}$ and $I_1^{(b,m,p)}$, we wanted to find out relations among them using integration by parts. At a certain stage of our process we had to obtain primitives of some functions. The integral we were trying to solve was $I(w) = \int w^{-5}(w^2 - 3w + 1) \log(w+1) \exp(w - w^{-1}) dw$. The built-in function `Integrate` of Mathematica, Version 4.0 produces the solution $I(w) = \exp(w - w^{-1})/w[\log(w+1)/w^2 - 1]$. This result is wrong as the derivative $I'(w)$ (of the second expression) is not the integrand of the integral $I(w)$. It means that Mathematica has a bug in the function `Integrate`. We have checked that Versions 2 and 3 are not able to evaluate $I(w)$ in terms of known functions.

m is zero, that is, for functions $\mathcal{V}_i^{(j,k,0)}$. In this situation the argument of the node is not present in the function \mathcal{R}_i and the powers of w are integers.

At this point we must emphasize the problems which can arise from the application of Propositions 4.1 and 4.2. Concretely, if for a certain order i of the Lie process one arrives at a generating function like (21) and the passage to order $i+1$ is needed, it is important to know whether the Poisson bracket $\{\mathcal{H}_1^{(j-1)}, \mathcal{W}_i\}$ is a function like (16). Indeed, when the exponential part of the basic integrals (22) and (23) does not vanish \mathcal{W}_i does not admit an easy expression. This is related with the appearance of nonexact resonance or approximate at that order, that is, for some regions in Δ_D for which the term $T_i^{(l,m)}$ of Eq. (18) is almost null and therefore the Poisson bracket $\{\mathcal{H}_0, T_i^{(l,m)}\}$ vanishes but $m \neq 0$ and $\Omega \neq 0$. Then, in general, the intermediate Hamiltonian \mathcal{H}_{i+1} is not of the type (16). Thus, Proposition 4.2 cannot be applied at order $i+1$ and the process must be stopped at order i . Then, although the Lie transformation is built only up to order i both the normal form \mathcal{K} and the generating function \mathcal{W} are obtained in closed form up to order i . This feature is in contrast to the procedure of expanding formulæ in Fourier series of ℓ and in power series of e as with our approach the expressions are valid for all values $e \in (0, 1)$ (and indeed in $[0, 1)$ as we shall explain in Section 6).

However, in some cases the perturbation at a certain order i is independent of the argument of the node; on other occasions the primary body does not rotate (e.g., $\Omega \equiv 0$). In such circumstances it is still possible to reach order $i+1$ even if the node is present at \mathcal{H}_{i+1} . Moreover it may be possible to reach any order (apart from the limitations due to the algebraic manipulator and the computer one uses) as \mathcal{W} is analytic for any order i as the polylogarithmic function of argument z_E is analytic for E in $[0, 2\pi)$ (see details and examples of this in Refs. [38, 39]).

In the general situation $I_0^{(b,m)}$ and $I_1^{(b,m,p)}$ must be considered as special functions with complex argument. They can be related with a generalization of the incomplete gamma function as we show in the next section.

4.3. A Generalization of the Generalized Incomplete Gamma Function

The incomplete gamma functions are defined through the integrals

$$\gamma(\alpha, x) = \int_0^x t^{\alpha-1} \exp(-t) dt, \quad \Gamma(\alpha, x) = \int_x^\infty t^{\alpha-1} \exp(-t) dt \quad (24)$$

for $\text{Re } \alpha > 0$. These functions were first investigated by Legendre. If $\Gamma(\alpha)$ denotes the gamma function (observe that $\Gamma(\alpha) = \Gamma(\alpha, 0)$), the three functions are related by means of the formula $\Gamma(\alpha) = \gamma(\alpha, x) + \Gamma(\alpha, x)$. Many properties on these gamma functions and relations with other functions are given in [2, 24].

Chaudhry and Zubair [8] have introduced the functions

$$\gamma(\alpha, x; a) = \int_0^x t^{\alpha-1} \exp(-t - a/t) dt,$$

$$\Gamma(\alpha, x; a) = \int_x^\infty t^{\alpha-1} \exp(-t - a/t) dt$$

with $x > 0$, $a \in \mathbf{R}$, and $\operatorname{Re} \alpha > 0$. These integrals are generalizations of the incomplete gamma functions (24). They appear in the resolution of the heat equation with time-dependent boundary conditions and in the theory of probability.

The decomposition theorem for $\gamma(\alpha, x; a)$ and $\Gamma(\alpha, x; a)$ states that if $K_p(q)$ is the modified Bessel function of the second kind with parameter p and argument q , then $\gamma(\alpha, x; a) + \Gamma(\alpha, x; a) = 2a^{\alpha/2} K_\alpha(2\sqrt{a})$ for $a > 0$. Thus, it is enough to study either $\gamma(\alpha, x; a)$ or $\Gamma(\alpha, x; a)$. An interesting relation is the recurrence:

$$\Gamma(\alpha + 1, x; a) = \alpha \Gamma(\alpha, x; a) + a \Gamma(\alpha - 1, x; a) + x^\alpha \exp(-x - a/x). \quad (25)$$

Closed form solutions of $\Gamma(\alpha, x; a)$ exist for $x = 0$ and for $\alpha = -\frac{1}{2}, \frac{1}{2}, \frac{3}{2}, \dots$. References [8, 9] include many details on the use of these incomplete gamma functions, their fundamental properties, and some asymptotic expansions and graphs. See also Ref. [7].

Now, we relate $I_0^{(b,m)}$ with the generalized incomplete gamma functions. After making in (22) the change of variable $t = -m\Omega e w/(2n)$ we arrive at the identity

$$I_0^{(b,m)}(z_E, L, G; \mu, \Omega) = \left(-\frac{m\Omega e}{2n}\right)^{b+1} \left[\Gamma\left(b+1, -\frac{m\Omega e z_E}{2n}; -\frac{m^2 \Omega^2 e^2}{4n^2}\right) - \Gamma\left(b+1, -\frac{m\Omega e}{2n}; -\frac{m^2 \Omega^2 e^2}{4n^2}\right) \right].$$

So, with the aid of formula (25) one can relate $I_0^{(b,m)}$ with $I_0^{(b-1,m)}$ and $I_0^{(b-2,m)}$. Specifically, if $b \geq 1$ one has

$$I_0^{(b,m)}(z_E, L, G; \mu, \Omega) = \frac{2n}{m\Omega e} \left\{ z_E^b \exp\left[\frac{m\Omega e}{2n}(z_E - z_E^{-1})\right] - \frac{m\Omega e}{2n} I_0^{(b-2,m)}(z_E, L, G; \mu, \Omega) - b I_0^{(b-1,m)}(z_E, L, G; \mu, \Omega) \right\}, \quad (26)$$

whereas if $b < -1$, $I_0^{(b-2, m)}$ has to be written in terms of $I_0^{(b-1, m)}$ and $I_0^{(b, m)}$. Hence, it is enough to consider $I_0^{(b, m)}$ with $b \in (-1, 1]$ because when b is outside this interval, we can use (25) to bring b to $(-1, 1]$.

However, we cannot relate $I_1^{(b, m, p)}$ with the incomplete gamma functions and use of another complex function must be done so as to obtain recurrent expressions such as (25). We define a complex integral depending on five real parameters through

$$I_{b, c, d, s, p}(z) = \int_1^z \frac{w^b}{(cw + d)^p} \exp[s(w - w^{-1})] dw, \quad (27)$$

with b, c , and d being real coefficients whereas $p \in \mathbb{Z}^+ \cup \{0\}$ and s and z are in \mathbb{C} . If $c = 0$ then d cannot be zero and if $c \neq 0$ then $-d/c$ cannot belong to the path connecting 1 with z . Furthermore, the function is considered exclusively for convergent values of the integral. It is not hard to prove how integrals (22) and (23) are special cases of function (27). If one adjusts the coefficients of $I_0^{(b, m)}$ and $I_1^{(b, m, p)}$ with those of $I_{b, c, d, s, p}$, the relations are:

- (i) $I_0^{(b, m)}(z_E, L, G; \mu, \Omega) = I_{b, 0, 1, \frac{m\Omega e}{2n}, 0}(z_E);$
- (ii) $I_1^{(b, m, p)}(z_E, L, G; \mu, \Omega) = I_{b, 1+\eta, -e, \frac{m\Omega e}{2n}, p}(z_E).$

Note that for $I_0^{(b, m)}$, $c \equiv 0$ and $d \equiv 1$. For $I_1^{(b, m, p)}$ the quotient $-d/c \equiv (1+\eta)/e$ is a real expression in $(1, +\infty)$ —circular orbits have been excluded from Δ_D . Thus, in the two cases, $-d/c$ does not touch the unit circle $|z_E| = 1$. The expression $cw + d$ does not vanish and the condition for the existence of the integral (27) is fulfilled. Therefore, the function (27) covers the two basic integrals used to obtain \mathcal{W}_i . We need to give a recursive relation for the integrals $I_{b, c, d, s, p}$.

First of all $\gamma(\alpha, x; a)$ and $\Gamma(\alpha, x; a)$ are related with (27) through

$$\begin{aligned} \gamma(\alpha, x; a) &= (-\sqrt{-a})^\alpha \left[I_{\alpha-1, 0, 1, \sqrt{-a}, 0} \left(-\frac{x}{\sqrt{-a}} \right) - I_{\alpha-1, 0, 1, \sqrt{-a}, 0}(0) \right], \\ \Gamma(\alpha, x; a) &= 2a^{\alpha/2} K_\alpha(2\sqrt{a}) + (-\sqrt{-a})^\alpha \left[I_{\alpha-1, 0, 1, \sqrt{-a}, 0}(0) \right. \\ &\quad \left. - I_{\alpha-1, 0, 1, \sqrt{-a}, 0} \left(-\frac{x}{\sqrt{-a}} \right) \right], \end{aligned}$$

provided that $a \neq 0$. Henceforth we can think of $I_{b, c, d, s, p}$ as a new generalization of the incomplete gamma function of which $\gamma(\alpha, x; a)$ and $\Gamma(\alpha, x; a)$ are special cases.

The integral $I_{b,c,d,s,p}$ admits a recurrence with respect to the integer parameter p . By applying parts to $I_{b,c,d,s,p}$, one arrives at the relation

$$\begin{aligned} I_{b,c,d,s,p}(z) = & \frac{1}{c(1-p)} \left\{ \frac{z^b \exp[s(z-z^{-1})]}{(cz+d)^{p-1}} - \frac{1}{(c+d)^{p-1}} \right. \\ & - \frac{s}{c^2} I_{b-2,c,d,s,p-1}(z) + \frac{2ds-bc}{c^2} I_{b-2,c,d,s,p-2}(z) \\ & \left. + \frac{bcd-(c^2+d^2)s}{c^2} I_{b-2,c,d,s,p-3}(z) \right\}, \end{aligned} \quad (28)$$

for $p \geq 2$. When $c \equiv 0$, $I_{b,0,d,s,p}(z) = I_{b,0,1,s,0}(z)/d^p$ and (27) does not apply. Besides, $c+d \neq 0$ as we consider the quotient $-d/c$ outside the path connecting 1 with z . Finally, the limit integral $I_{b-2,c,d,s,-1}$ must be understood as the sum $I_{b,c,d,s,-1}(z) = cI_{b+1,0,1,s,0}(z) + dI_{b,0,1,s,0}(z)$.

Now we have to apply (25) for $I_0^{(b,m)}$ and (27) for $I_1^{(b,m,p)}$ in order to arrive at the basic functions:

$$\begin{aligned} I_0^{(b,m)}(z_E, L, G; \mu, \Omega) &= \int_1^{z_E} w^b \exp \left[\frac{m\Omega e}{2n} (w-w^{-1}) \right] dw, \quad b \in (-1, 1], \\ I_1^{(b,m,1)}(z_E, L, G; \mu, \Omega) &= \int_1^{z_E} \frac{w^b \exp \left[\frac{m\Omega e}{2n} (w-w^{-1}) \right]}{(1+\eta)w-e} dw, \quad b \in \mathbf{R}. \end{aligned} \quad (29)$$

In the two cases the path of integration must be taken as a partial (or the entire) circumference $|z_E|=1$. Notice that for $I_1^{(b,m,1)}$ it has not been possible to reduce the interval of existence for the parameter b .

It is time now to relate (29) with \mathcal{W} . We do it as follows.

PROPOSITION 4.3. *The generating function $\mathcal{W}(\ell, g, h, L, G, H)$ given by*

$$\mathcal{W} = \mathcal{W}_1 + \varepsilon \mathcal{W}_2 + \frac{\varepsilon^2}{2} \mathcal{W}_3 + \cdots + \frac{\varepsilon^{M-1}}{(M-1)!} \mathcal{W}_M,$$

obtained by means of formula (21), defines an analytic function in the subset of \mathbf{R}^6 :

$$\Delta_D = [0, 2\pi) \times [0, 2\pi) \times [0, 2\pi) \times (0, +\infty) \times (0, L) \times (-G, G).$$

Proof. It is enough to prove that integrals $I_0^{(b,m)}$ and $I_1^{(b,m,1)}$ defined by (29) are analytic functions with respect to their three arguments z_E , L , and G in an adequate subset of Δ_D (observe that g , h , and H have fixed

values). We start then by proving the analyticity with respect to $z_E = \exp(\imath E)$ with $E \in [0, 2\pi)$. Thus we have to calculate the derivatives of $I_0^{(b,m)}$ and $I_1^{(b,m,1)}$ with respect to z_E . They are:

$$\frac{\partial I_0^{(b,m)}}{\partial z_E}(z_E, L, G; \mu, \Omega) = z_E^b \exp\left[\frac{m\Omega e}{2n}(z_E - z_E^{-1})\right],$$

$$\frac{\partial I_1^{(b,m,1)}}{\partial z_E}(z_E, L, G; \mu, \Omega) = \frac{z_E^b \exp\left[\frac{m\Omega e}{2n}(z_E - z_E^{-1})\right]}{(1+\eta)z_E - e}.$$

Taking into account that $b = q - m\Omega/n$ with $q \in \mathbb{Z}$ and that $z_E = \exp(\imath E)$ and making use of Kepler equation $\ell = E - e \sin E$, the latter partial derivatives become

$$\frac{\partial I_0^{(b,m)}}{\partial z_E}(E, L, G; \mu, \Omega) = \exp[\imath(qE + m\Omega\ell/n)],$$

$$\frac{\partial I_1^{(b,m,1)}}{\partial z_E}(E, L, G; \mu, \Omega) = \frac{\exp[\imath(qE + m\Omega\ell/n)]}{(1+\eta)\exp(\imath E) - e}.$$

Now, these expressions are well defined for any value of $E, \ell \in [0, 2\pi)$ and for all $e \in (0, 1)$. Moreover, they are analytic functions in E . Hence, $I_0^{(b,m)}$ and $I_1^{(b,m,1)}$ are analytic functions in the argument $z_E = \exp(\imath E)$.

Next, the partial derivatives of $I_0^{(b,m)}$ and $I_1^{(b,m,1)}$ with respect to $L \in (0, +\infty)$ and $G \in (0, L)$, that is, the expressions $\partial I_0^{(b,m)}/\partial L$, $\partial I_0^{(b,m)}/\partial G$, $\partial I_1^{(b,m,1)}/\partial L$ and $\partial I_1^{(b,m,1)}/\partial G$ must be computed taking into account that E depends on both L and G since E is related to ℓ through the eccentricity (recall that $e = (1 - G^2/L^2)^{1/2}$). These derivatives are well defined functions in \mathcal{A}_D and the partial derivatives of any order can be obtained from them. Although these partial derivatives have, in general, logarithmic expressions in some of their integrands, one has to take into account that with the change $w = \exp(\imath\kappa)$ (with $\kappa \in [0, 2\pi)$), the logarithm disappears and the integrands yield analytic expressions.

These partial derivatives have analytic integrands and the corresponding integrals are well defined in the sense that no singularity is introduced by the integration path, which in the variable κ is the real interval $[0, 2\pi)$. Thus, $I_0^{(b,m)}$ and $I_1^{(b,m,1)}$ are analytic functions with respect to z_E (i.e., to E), L , and G . Finally, \mathcal{W} is analytic in \mathcal{A}_D . ■

Note that $\mathcal{W}_i \in \mathcal{C}_i$ and, according to the above proposition, we can take each \mathcal{C}_i as a space of smooth functions defined on \mathcal{A}_D . Besides, the sets \mathcal{B}_i where we have taken each \mathcal{X}_i are defined on \mathcal{A}_D . They are indeed the

smooth subspaces of \mathcal{A}_i where ℓ is ignorable (if L is the new integral) or h is ignorable (if H the new integral).

The relevant features of this section are collected in the next theorem.

THEOREM 4.1. *The three-degree-of-freedom Hamiltonian system $\mathcal{H} = \mathcal{H}_0 + \mathcal{H}_1$ defined by*

$$\mathcal{H}_0(L, H) = -\frac{\mu^2}{2L^2} - \Omega H \quad (30)$$

and

$$\mathcal{H}_1(\ell, g, h, L, G, H) = \sum_{\substack{(j, l, m) \in \mathbb{Z}^3 \\ k \in \mathbb{Z}^+ \cup \{0\}}} \left(\frac{r}{\alpha}\right)^j \left(\frac{R}{\beta}\right)^k \mathcal{F}^{(l, m)}(g, h, L, G, H), \quad (31)$$

with

$$\begin{aligned} \mathcal{F}^{(l, m)}(g, h, L, G, H) &= \mathcal{C}^{(l, m)}(L, G, H) \cos(lg + mh) \\ &\quad + \mathcal{S}^{(l, m)}(L, G, H) \sin(lg + mh) \end{aligned}$$

can be transformed into a system defined by the Hamiltonian

$$\mathcal{K} = \mathcal{K}_0 + \varepsilon \mathcal{K}_1 + \frac{\varepsilon^2}{2} \mathcal{K}_2 + \cdots + \frac{\varepsilon^M}{M!} \mathcal{K}_M$$

by means of a generating function

$$\mathcal{W} = \mathcal{W}_1 + \varepsilon \mathcal{W}_2 + \frac{\varepsilon^2}{2} \mathcal{W}_3 + \cdots + \frac{\varepsilon^{M-1}}{(M-1)!} \mathcal{W}_M,$$

such that:

(a) *The changes of variables to pass from \mathcal{H} to \mathcal{K} and vice versa are given by $\mathbf{x} = \mathbf{X}(\mathbf{y}; \varepsilon)$ and $\mathbf{y} = \mathbf{Y}(\mathbf{x}; \varepsilon)$, respectively. Vector \mathbf{x} denotes old variables, i.e., the six-tuple (ℓ, g, h, L, G, H) whereas the new variable \mathbf{y} stands for $(\ell', g', h', L', G', H')$ (primes are dropped to simplify notation).*

(b) *Changes $\mathbf{y} = \mathbf{Y}(\mathbf{x}; \varepsilon)$ and $\mathbf{x} = \mathbf{X}(\mathbf{y}; \varepsilon)$ are built, respectively, following formulae (11) and (12), truncating the series at order M in ε . They are global formal symplectic \mathcal{C}^M -diffeomorphisms in Δ_D as \mathcal{W} is analytic in that domain.*

(c) *Hamiltonian \mathcal{K} is the normal form of \mathcal{H} and, in particular, $\mathcal{K}_0 \equiv \mathcal{H}_0$. It defines a dynamical system of two degrees of freedom for which either L or H becomes a (formal) integral of it. Thus, in the rotating situation $\Omega \neq 0$, two different normal forms can be constructed leading, therefore, to the analysis of the original system \mathcal{H} from two different points of view.*

(d) *The order $M \geq 1$ reached depends on the type of perturbation \mathcal{H}_1 and of the frequency Ω . The process of the Lie transformation must be stopped when the integrals $I_0^{(b,m)}$ and $I_1^{(b,m,p)}$ (with $\Omega \neq 0$ and $m \neq 0$) appear in the generating function. For those terms independent of the node (e.g., terms $P^{j,k,l,m}$ with $m \equiv 0$) or for nonrotating systems \mathcal{W} and \mathcal{H} are analytic functions and the computations can be pushed up to any order.*

Proof. It follows Propositions 4.1, 4.2, and 4.3 together with other results given through the section. ■

Note that the normal form is constructed in the domain Δ_D where Delaunay variables are properly defined. However, it is possible to extend the existence domain of \mathcal{H} by including (noncollision) rectilinear, equatorial, and circular trajectories. This is done using an argument of continuation of the analytic function \mathcal{H} , but we need special coordinates well defined for the orbits mentioned. This will be done in Section 6.

5. NORMAL FORMS FOR NON-COMPARABLE ROTATING AND TWO-BODY EFFECTS

5.1. Slow Rotations

Perturbations of rotating Keplerian systems with slow rotation can be considered as a particular case of the general situation treated in Section 4. Now, in Delaunay variables, $\mathcal{H}_0(L) = -\mu^2/(2L^2)$ whereas $-\Omega H$ goes to the first order. Besides, the perturbation \mathcal{P} has to be placed at least at first order.

This situation is rather typical in the planar or spatial three-body problem when dealing with Hill-type orbits (orbits for which the infinitesimal particle is very near one of the primaries); see [34]. In the context of the artificial satellite theory, this case appears for satellites orbiting at low altitude [40]. For these satellites, the ratio Ω/n (e.g., the quotient between the rotation of the main planet, usually the Earth, and the mean motion of the satellite) is small. Also this situation occurs when $\Omega/n \approx 1$ (geostationary satellites) if the orbits are almost polar; i.e., $H \approx 0$. In both cases the term $-\Omega H$ is put at first order whereas the zonal and tesseral harmonic terms are placed at higher orders and a Delaunay normalisation, (cf. [17]) is applied to calculate the normal form by making L an integral out of the normal form. Kummer [26, 27] analysed the three-dimensional lunar problem and a certain class of perturbations of the rotating Kepler system whereas Cushman and Sadovskii [14] dealt with the motion of an electron in a strongly ionized hydrogen atom. The fact that the rotating term in the

lunar problem is much smaller than the Kepler Hamiltonian and that the hydrogen atom is strongly ionized in the second case permit us to place the Coriolis term at first order whereas the two-body component of \mathcal{H}_0 remains at zeroth order.

Now, the Lie operator associated to \mathcal{H}_0 in Delaunay variables is $\mathcal{L}_{\mathcal{H}_0} = n\partial(\cdot)/\partial\ell$ and, for each $i \geq 1$, the homology equation (10) this time is:

$$n \frac{\partial \mathcal{W}_i}{\partial \ell} + \mathcal{K}_i = \tilde{\mathcal{H}}_i, \quad (32)$$

Now, the normal form is the Hamiltonian $\mathcal{K} = \mathcal{H}_0 + \varepsilon \mathcal{K}_1 + (\varepsilon^2/2) \mathcal{K}_2 + \dots + (\varepsilon^M/M!) \mathcal{K}_M$ such that we have introduced an integral through the transformation. The usual way of achieving this is by choosing $\mathcal{J} = \mathcal{H}_0$ or, in other words, by forcing to \mathcal{K} to be independent of ℓ . Thus, \mathcal{K} will define a two-degree-of-freedom system in g, h, G , and H . In this way, we will have that the Poisson bracket $\{\mathcal{K}_i, \mathcal{H}_0\} = 0$ for $i = 0, \dots, M$. This is completely equivalent to the so-called Delaunay normalisation in two dimensions. Thus, what goes on in this section is valid for nonrotating systems ($\Omega \equiv 0$) or for two-dimensional systems of the type $\mathcal{H} \equiv \mathcal{H}(\ell, g, L, G)$. For the latter situation, the problem has already been solved, although we present an alternative way of determining \mathcal{K} and \mathcal{W} . See details in Refs. [17, 36]. Extensions of these results appear in [38].

Let us consider perturbations such as (13) and (14). We can put the latter in the perspective of Section 4. We start by computing \mathcal{K}_i . Given a term $P_i^{(j,k,l,m)}$ as in (17), we calculate $P_i^{(j,k,l,m)} = \frac{1}{2\pi} \int_0^{2\pi} P_i^{(j,k,l,m)} d\ell$. So $P_i^{(j,k,l,m)}$ is the contribution of $P_i^{(j,k,l,m)}$ to \mathcal{K}_i . The latter integral is equivalent to that of Section 4 when we chose $\mathcal{J} = -\mu^2/(2L^2)$. The best way of solving the integral consists in finding first a primitive of $P_i^{(j,k,l,m)}$ (which is equivalent to obtain $Q_i^{(j,k,l,m)}$) and applying then Barrow's rule. A primitive can be obtained following the strategies we show just below.

Thus, for the calculation of $Q_i^{(j,k,l,m)}$ (equivalently the calculation of \mathcal{W}_i), Propositions 4.1 and 4.2 apply although in a simplified manner. Indeed, after making the change of variable $z_E = \exp(\iota E)$ one arrives at expressing \mathcal{W}_i as in (21) but considering $\Omega \equiv 0$. The reason is that now $-\Omega H$ is at first order and this term does not affect the resolution of the homology equation (32). Specifically, once \mathcal{K}_i is obtained, the solution of the homology equation (32), after changing from ℓ to z_E , is

$$\begin{aligned} \mathcal{W}_i(z_E, g, h, L, G, H) = & \sum_{\substack{(j,l,m) \in \mathbb{Z}^3 \\ k \in \mathbb{Z}^+ \cup \{0\}}} \mathcal{W}_i^{(j,k,l,m)}(g, h, L, G, H) \\ & \times \int_1^{z_E} \mathcal{V}_i^{(j,k)}(w, L, G) dw \end{aligned} \quad (33)$$

where

$$\begin{aligned}\mathcal{W}_i^{(j,k,l,m)}(g,h,L,G,H) &= \frac{(-1)^j l^{1+k} e^k \mathcal{G}_i^{(l,m)}(L,G,H)}{2^{1+j} \alpha^j \beta^k \mu^{2+j-k} L^{-3-2j+k}} \\ &\quad \times \exp[l(lg+mh)], \\ \mathcal{V}_i^{(j,k)}(w,L,G) &= \frac{(w^2-1)^k (ew^2-2w+e)^{1+j-k}}{w^{2+j}}.\end{aligned}$$

Note that $\mathcal{V}_i^{(j,k)}$ is a rational function in w , $j \in \mathbf{Z}$ and $k \in \mathbf{Z}^+ \cup \{0\}$. Recurrent relations of $I_0^{(b)}$ and $I_1^{(b,m)}$ analysed in Section 4 can be used (with $m=0$). Thus, the integrals to be solved are:

$$\begin{aligned}I_0^{(q)}(z_E, L, G) &= \int_1^{z_E} w^q dw, \\ I_1^{(q)}(z_E, L, G) &= \int_1^{z_E} \frac{w^q}{(1+\eta)w-e} dw, \\ I_2^{(q)}(z_E, L, G) &= \int_1^{z_E} \frac{w^q}{-ew+1+\eta} dw.\end{aligned}\tag{34}$$

In the three cases the integration path must be taken as a part (or the entire) circumference $|z_E|=1$ and $q \in \mathbf{Z}$.

As a first attempt to solve (34), it is possible to give directly the expressions of $I_0^{(q)}$ and $I_1^{(q)}$ in terms of z_E . In fact these results are always combinations of positive and negative powers of z_E and logarithms containing z_E . Nevertheless we can follow an alternative route. Notice that these two integrals appear when the condition $1+j-k < 0$ occurs in the computation of \mathcal{W}_i . However, it is possible to make a new change of variables to avoid the presence of logarithms and arctangent functions caused by the explicit solutions of $I_1^{(q)}$. The idea consists in using the true anomaly through the change of variable $z_f = \exp(if)$. It was proposed by Hansen (see Smart [42, p. 31]). In [36] it has been used to obtain closed form formulæ for $\int (\varphi R/r) d\ell$ and $\int (\log(a\eta^2/r))/r^2 d\ell$.

Variables z_E and z_f are related by

$$z_E = \frac{(1+\eta)z_f + e}{ez_f + 1 + \eta}, \quad z_f = \frac{(1+\eta)z_E - e}{-ez_E + 1 + \eta}, \quad dz_E = \frac{2\eta(1+\eta)}{(ez_f + 1 + \eta)^2} dz_f.\tag{35}$$

The passage from z_E to z_f transforms the integral of Eq. (33) into

$$\int_1^{z_f} \mathcal{V}_i^{(j,k)}(u, L, G) du$$

with

$$\mathcal{V}_i'^{(j,k)}(u, L, G) = (2\eta)^{3+2j-k} \frac{(-u)^{1+j-k} (u^2-1)^k}{(eu^2+2u+e)^{2+j}}.$$

This implies that the integrand $\mathcal{V}_i'^{(j,k)}$ can be decomposed into positive and negative powers of u if and only if $j \leq -2$, independent of the value of k . Thus, when $j \geq -1$ we shall employ z_E whereas when $j \leq -2$ the variable z_f works better.

Let us analyse the above with more detail. We shall say that the pair (j, k) verifies the D'Alembert characteristic (for slow or nonrotating Kepler problems) when both j and k are integers satisfying either (i) $1+j-k \geq 0$ and $k \geq 0$ or (ii) $1+j-k < 0$, $k \geq 0$, and $j \leq -2$. Thus, if for a term $P_i^{(j,k,l,m)}$, the pair (j, k) verifies (i) we shall use z_E to obtain its anti-image and if it verifies (ii) we shall work with z_f . However, the introduction of logarithmic terms cannot be avoided. This comes from two facts: (a) the D'Alembert characteristic does not hold (for instance $j=1$ and $k=3$); (b) the D'Alembert characteristic holds but one has to calculate the primitive of z_E^{-1} or z_f^{-1} . In both situations we still can calculate the corresponding integrals. This justifies the presence of $\log(z_E)$, $\log(z_f)$, and other combinations such as the equation of the centre, which is a combination of logarithms, if one writes it in terms of z_E with the aid of (20). Thence, the dilogarithm appears at order $i+1$ when handling primitives of functions whose integrands contain logarithms of z_E (or of z_f) multiplied by rational expressions of z_E (or of z_f); see cf. [36] for more details. Besides, it is possible to reach higher order terms and handle the formulæ containing the dilogarithm in terms of polylogarithms as it is shown in [38, 39].

In order to calculate both \mathcal{K} and \mathcal{W} it is convenient to use z_E and z_f instead of r , R , f , and E (these latter variables are the ones used in the algorithms of Delaunay normalisation). The reason is that from a computational point of view, it is much faster to use powers of monomials z_E and z_f than trigonometric expressions of the anomalies and powers of r and R . Note that the calculation of the integrals involving either z_E or z_f is performed straightforwardly. Besides, whether $j \leq -2$ the standard techniques based on regularization do not attain to express the perturbation as a polynomial in \mathbf{R}^2 or \mathbf{R}^4 ; consequently, Birkhoff normalisation cannot be performed and one is advised to use the variable z_f . All this shows the advantages of using the present approach when dealing with slow rotating Keplerian systems.

Now, the calculation of $P_i'^{(j,k,l,m)}$ can be done after computing the primitives $\int P_i^{(j,k,l,m)} d\ell$ in terms of z_E and/or z_f , passing back to E and f and applying Barrow's law with the extreme points $\ell=f=E=0$ and $\ell=f=E=2\pi$. This completes the determination of \mathcal{K}_i .

It is remarkable that the change $z_f = \exp(\iota f)$ could have been done for the case studied in Section 4. However, the trouble when Ω and m do not vanish is that the exponent of the integrands appearing in $I_0^{(b,m)}$ and $I_1^{(b,m,p)}$ has a more involved expression as a function of z_f . This feature makes the recurrent relations among the integrals much more complicated. Thus, we preferred to employ z_E through the whole Section 4. It could be possible to make the change of variable directly in formula (16) for those terms with $j \leq -2$. Thence we would avoid using first z_E and then change to z_f . So we necessitate putting the variables related with ℓ in terms of f . It yields:

$$\begin{aligned}\ell &= \iota \left\{ \frac{e\eta(z_f^2 - 1)}{ez_f^2 + 2z_f + e} + \log \left[\frac{ez_f + 1 + \eta}{(1 + \eta)z_f + e} \right] \right\}, \\ f &= -\iota \log(z_f), \quad E = \iota \log \left[\frac{ez_f + 1 + \eta}{(1 + \eta)z_f + e} \right], \\ r &= \frac{2G^2 z_f}{\mu(ez_f^2 + 2z_f + e)}, \quad R = \frac{\mu e(1 - z_f^2)}{2Gz_f}.\end{aligned}\tag{36}$$

Thus, we could replace in a term $P_i^{(j,k,l,m)}$ with $j \leq -2$, r and R (also the angles f or E if they appear explicitly in the perturbation) conveniently and use the variable z_f to perform the calculations of \mathcal{K} and \mathcal{W} .

5.2. Very Fast Rotations

This time $\mathcal{H}_0(H) = -\Omega H$ and $-\mu^2/(2L^2)$ is placed at first order. This case appears when dealing with orbits close to infinity in the restricted three-body problem [34] (comet-type orbits). The Coriolis force dominates and the next most important force is like the Keplerian problem with both primaries at the origin.

The Lie operator associated to \mathcal{H}_0 in Delaunay variables is $\mathcal{L}_{\mathcal{H}_0} = -\Omega \partial(\cdot)/\partial h$ and, for each $i \geq 1$, the homology equation (10) reads:

$$-\Omega \frac{\partial \mathcal{W}_i}{\partial h} + \mathcal{K}_i = \tilde{\mathcal{H}}_i,\tag{37}$$

The normal form is $\mathcal{K} = \mathcal{K}_0 + \varepsilon \mathcal{K}_1 + (\varepsilon^2/2) \mathcal{K}_2 + \dots + (\varepsilon^M/M!) \mathcal{K}_M$ in which the angle h is not present; that is, \mathcal{K} is axially symmetric with respect to the axis z . In this manner, the Poisson bracket $\{\mathcal{K}_i, \mathcal{H}_0\} = 0$ for $i = 0, \dots, M$.

The Lie process which must be carried out in this case is much easier than the two previous cases (Sections 4 and 5.1). The reason is that a perturbation term, say $P_i^{(j,k,l,m)}$, depends on the argument of the node in a simple trigonometric manner and it did not occur with the perturbation

terms and its dependence on the mean anomaly. Recall that $P_i^{(j,k,l,m)}$ is given by formula (17). Then, the antiimage of $P_i^{(j,k,l,m)}$ through $\mathcal{L}_{\mathcal{H}_0}$ is obtained after calculating the integral $P_i'^{(j,k,l,m)} = \frac{1}{2\pi} \int_0^{2\pi} P_i^{(j,k,l,m)} dh$, straightforwardly, whereas $Q_i^{(j,k,l,m)} = \frac{1}{\Omega} \int [\mathcal{P}_i'^{(j,k,l,m)} - P_i^{(j,k,l,m)}] dh$. Note that $P_i'^{(j,k,l,m)}$ is completely independent of the argument of the node and $Q_i^{(j,k,l,m)}$ is a periodic function in that angle. Therefore, \mathcal{H} will define a system with two degrees of freedom in the variables ℓ , g , L , and G .

6. REDUCTIONS

6.1. Invariants Associated to the Reductions

The normal form transformations in Sections 4 and 5 lead to the introduction of an integral. The appearance of this integral allows the initial system to be reduced by one degree of freedom; i.e., the normal form \mathcal{H} defines a dynamical system of two degrees of freedom.

For each normal form transformation, we have to describe the phase space where \mathcal{H} is defined. This phase space has dimension four. It is constructed according to the integral introduced in the Lie transformation. Therefore, two different phase spaces are considered for perturbed Keplerian systems: one associated to the integral L and the other associated to H . One should realize that the integral \mathcal{I} we take (either $\mathcal{I} = \mathcal{H}_K$ or $\mathcal{I} = \mathcal{H}_C$) represents a maximally superintegrable system, that is, it possesses five independent integrals of motion. Thence, we can employ, after fixing the value of one of the five, the other four to parameterize the reduced phase space. This is why the phase space out of the reduction process has dimension four. However, this is not the general situation for central potentials, in which only four independent integrals can be constructed as it is shown for the isochrone model in [47].

Note that \mathcal{H}_K and \mathcal{H}_C define complete Hamiltonian vector fields; see cf. [1]. Therefore their flows define group actions and hence the reduction theorem can be applied in both cases; see [3, 31] for more details.

6.1.1. The Action L Becomes an Integral

The integrals associated to L —or to $-\mu^2/(2L^2)$ —are the functions which are constants on the solutions on the system defined by \mathcal{H}_K . All these integrals can be expressed as functions of L , the components of the angular momentum vector $\mathbf{G} = (G_1, G_2, G_3)$ (note that $G_3 = H$), and the Laplace vector $\mathbf{A} = (A_1, A_2, A_3)$; e.g., the vector defined as

$$\mathbf{A} = \frac{1}{\mu} (\mathbf{X} \times \mathbf{G}) - \frac{\mathbf{x}}{\|\mathbf{x}\|};$$

see cf. [13] for more details. Observe that $\|\mathbf{G}\| = G$, $\|\mathbf{A}\| = e$, and $\mathbf{G} \cdot \mathbf{A} = 0$.

In [12, 13] the mapping

$$\rho: \mathbf{R}^6 \setminus (\{\mathbf{0}\} \times \mathbf{R}^3) \rightarrow \mathbf{R}^6: (\mathbf{x}, \mathbf{X}) \mapsto (\mathbf{a}, \mathbf{b}) \equiv (\mathbf{G} + L\mathbf{A}, \mathbf{G} - L\mathbf{A})$$

is considered, with $\mathbf{a} = (a_1, a_2, a_3)$ and $\mathbf{b} = (b_1, b_2, b_3)$. Explicitly, the functions a_i and b_i can be given in terms of the coordinates \mathbf{x} and \mathbf{X} . However, we give their expressions in terms of the Delaunay variables:

$$\begin{aligned} a_1 &= G \sin h \sin I + Le \cos g \cos h - Le \sin g \sin h \cos I, \\ a_2 &= -G \cos h \sin I + Le \cos g \sin h + Le \sin g \cos h \cos I, \\ a_3 &= G \cos I + Le \sin g \sin I, \\ b_1 &= G \sin h \sin I - Le \cos g \cos h + Le \sin g \sin h \cos I, \\ b_2 &= -G \cos h \sin I - Le \cos g \sin h - Le \sin g \cos h \cos I, \\ b_3 &= G \cos I - Le \sin g \sin I. \end{aligned} \tag{38}$$

Note that $\cos I = H/G$ and $\sin I = (1 - H^2/G^2)^{1/2}$. Variables a_i and b_i are indeed the coordinates used to describe the reduced phase space as they are the functions associated to the vector fields generating the $SO(4)$ symmetry of $-\mu/(2L^2)$.

Now, fixing a value of $-\mu^2/(2L^2) < 0$, the product of the two-sphere

$$S_L^2 \times S_L^2 = \{(\mathbf{a}, \mathbf{b}) \in \mathbf{R}^6 \mid a_1^2 + a_2^2 + a_3^2 = L^2, b_1^2 + b_2^2 + b_3^2 = L^2\}, \tag{39}$$

is the phase space for Hamiltonian systems of Keplerian type independent of ℓ , that is, for Hamiltonians for which L is an integral. This result was first reported by Moser [35] using a regularization technique based on stereographic projections. Observe that $S_L^2 \times S_L^2$ is a smooth space and therefore the reduction is regular [1, 31]. Note that in two dimensions, the corresponding reduced phase space is $S_L^1 \times S_L^1 \cong T^2$.

From (38) it is easy to deduce that

$$\begin{aligned} 2G \sin h \sin I &= a_1 + b_1, \\ 2Le(\cos g \cos h - \sin g \sin h \cos I) &= a_1 - b_1, \\ -2G \cos h \sin I &= a_2 + b_2, \\ 2Le(\cos g \sin h + \sin g \cos h \cos I) &= a_2 - b_2, \\ 2G \cos I &= a_3 + b_3, \\ 2Le \sin g \sin I &= a_3 - b_3, \end{aligned}$$

which allows us to express the functions G , H , $\cos g$, $\sin g$, $\cos h$ and $\sin h$ in terms of \mathbf{a} and \mathbf{b} . Now, a Hamiltonian \mathcal{H} independent of ℓ can be written as a function of the invariants \mathbf{a} and \mathbf{b} and the constant $L > 0$; i.e., $\mathcal{H} \equiv \mathcal{H}(\mathbf{a}, \mathbf{b}; L)$. Note that the way in which the invariants appear in the Hamiltonian \mathcal{H} depends on each specific problem.

The functions a_i and b_i are the invariants associated to $S_L^2 \times S_L^2$. These elements together with the constraints $a_1^2 + a_2^2 + a_3^2 = L^2$ and $b_1^2 + b_2^2 + b_3^2 = L^2$ define the reduced phase space, so they are called generators or coordinates of the reduced phase space.

Since $2G = ((a_1 + b_1)^2 + (a_2 + b_2)^2 + (a_3 + b_3)^2)^{1/2}$, one has that $G = 0$ in $S_L^2 \times S_L^2$ if and only $a_1 + b_1 = a_2 + b_2 = a_3 + b_3 \equiv 0$, $a_1^2 + a_2^2 + a_3^2 = L^2$ and $b_1^2 + b_2^2 + b_3^2 = L^2$. These relations define a two-sphere $\mathcal{R}_L = \{(\mathbf{a}, -\mathbf{a}) \in \mathbf{R}^6 \mid a_1^2 + a_2^2 + a_3^2 = L^2\}$, and rectilinear trajectories could be taken into account.

Circular orbits can be analysed since they are connected by the condition $G = L$, which in terms of \mathbf{a} and \mathbf{b} are given by the three-dimensional set

$$\begin{aligned} \mathcal{C}_L = \{(\mathbf{a}, \mathbf{b}) \in \mathbf{R}^6 \mid a_1^2 + a_2^2 + a_3^2 = L^2, b_1^2 + b_2^2 + b_3^2 = L^2, \\ a_1 b_1 + a_2 b_2 + a_3 b_3 = L^2\}. \end{aligned}$$

Similarly, equatorial trajectories (they satisfy $G = |H|$) can be treated with the invariants as they are described by the two-dimensional set $\mathcal{E}_L = \{(\mathbf{a}, \mathbf{b}) \in \mathbf{R}^6 \mid a_1^2 + a_2^2 + a_3^2 = L^2, b_1 = -a_1, b_2 = -a_2, b_3 = a_3\}$.

The above shows how the introduction of the invariants extends the use of the Delaunay variables as we can include equatorial, circular, and rectilinear orbits. We need the Poisson brackets between the elements of \mathbf{a} and \mathbf{b} . They are those of Table I.

TABLE I
Poisson Brackets for the Invariants a_i and b_i

$\{ , \}$	a_1	a_2	a_3	b_1	b_2	b_3
a_1	0	$2a_3$	$-2a_2$	0	0	0
a_2	$-2a_3$	0	$2a_1$	0	0	0
a_3	$2a_2$	$-2a_1$	0	0	0	0
b_1	0	0	0	0	$2b_3$	$-2b_2$
b_2	0	0	0	$-2b_3$	0	$2b_1$
b_3	0	0	0	$2b_2$	$-2b_1$	0

Note. The invariants on the left must be put in the left side of the bracket, whereas the ones on the top are placed at the right side.

6.1.2. The Action H Becomes an Integral

The integrals associated to H —or to the single term $-\Omega H$ —are the constant functions on the solutions on the system defined by \mathcal{H}_C . From the theory of invariants it can be deduced that a set of generators of invariant polynomials for the S^1 -action is given by:

$$\begin{aligned} c_1 &= x^2 + y^2, & c_2 &= xX + yY, & c_3 &= z, \\ c_4 &= X^2 + Y^2, & c_5 &= xY - yX, & c_6 &= Z, \end{aligned} \quad (40)$$

collected in the six-dimensional vector \mathbf{c} . The components of \mathbf{c} satisfy:

$$c_1 c_4 = c_2^2 + c_5^2. \quad (41)$$

Making use of Eqs. (4) and (5) it can be possible to express \mathbf{c} as a combination of polar-nodal and Delaunay variables. However, one can identify c_5 with H . Fixing a value of H (with $|H| \leq G$), this integral H can be understood as an S^1 -action, or the action of the one-dimensional unitary group $U(1)$ over the space of coordinates and moments such that

$$\begin{aligned} \rho: S^1 \times (\mathbf{R}^6 \setminus (\{\mathbf{0}\} \times \mathbf{R}^3)) &\rightarrow \mathbf{R}^3 \times \mathbf{R}^3 \\ (R_h, (\mathbf{x}, \mathbf{X})) &\mapsto (R_h \mathbf{x}, R_h \mathbf{X}), \end{aligned}$$

where

$$R_h = \begin{pmatrix} \cos h & \sin h & 0 \\ -\sin h & \cos h & 0 \\ 0 & 0 & 1 \end{pmatrix} \quad (42)$$

with $0 \leq h < 2\pi$. In fact, as it is exposed in [1], the subgroup of the special orthogonal group $SO(3)$: $\mathcal{O}_{\mathbf{s}_3} = \{O \in SO(3) \mid O \mathbf{s}_3 = \mathbf{s}_3\} = \{R_h \mid 0 \leq h < 2\pi\}$ is diffeomorphic to S^1 .

This is a singular (or nonfree) action because there are nontrivial isotropy groups. The subspace $\{(0, 0, z) \mid z \in \mathbf{R}\}$ is invariant under all rotations around the axis z . Thus, the reduction due to the axial symmetry is singular, in contrast to the regular reduction obtained by doing L an integral, where all the isotropy groups were trivial. Then we have to apply a singular reduction treatment [3].

The reduced phase space is given now as the quotient space $\mathbf{R}^6/\rho = \mathbf{R}^6/(S^1 \times S^1)_H$ for a fixed value of H , that is,

$$\mathbf{R}^6/(S^1 \times S^1)_H = \{\mathbf{c} \in \mathbf{R}^6 \mid c_1 c_4 = c_2^2 + c_5^2, c_5 = H, c_1, c_4 \geq 0\}. \quad (43)$$

It is a four-dimensional space whose generators are the invariants \mathbf{c} defined by (40) with the constraint (41) and $c_5 = H$. In two dimensions $\mathcal{H}_0 = -\mu^2/(2L^2) - \Omega G$ and the reduced phase space after making $-\Omega G$ an integral out of the normal form turns out to be $\{\mathbf{c} \in \mathbf{R}^4 \mid c_1 c_2 = c_3^2 + c_4^2, c_4 = G\}$.

We give the Poisson brackets among the components of \mathbf{c} as they are useful to analyse a certain normal form $\mathcal{K}(\mathbf{c}; H)$. The list is given in Table II.

This time, \mathcal{K} can be written as a function of the invariants \mathbf{c} and the constant $|H| \leq G$ as a parameter; e.g., $\mathcal{K} \equiv \mathcal{K}(\mathbf{c}; H)$.

Rectilinear orbits can be considered in (43). Note that $G = 0$ if and only if $xY - yX = 0$, $xZ - zX = 0$, and $zY - yZ = 0$. Combining this with the constraints given in (43), we have that the space of rectilinear trajectories is a three-dimensional subset of $\mathbf{R}^6/(S^1 \times S^1)_H$ defined as $\mathcal{R}_H = \{\mathbf{c} \in \mathbf{R}^6 \mid c_5 = 0, c_1 c_4 = c_2^2, c_1 c_6 = c_2 c_3, c_2 c_6 = c_2 c_4\}$. So rectilinear orbits, excepting $\|\mathbf{x}\| = 0$, may be analysed in this context.

Circular and equatorial orbits can be treated without restrictions since the invariants \mathbf{c} are not derived from the Delaunay variables. So they should be treated with the invariants. For instance, equatorial trajectories are in the two-dimensional set $\mathcal{E}_H = \{\mathbf{c} \in \mathbf{R}^6 \mid c_1 c_4 = c_2^2 + c_5^2, c_3 = c_6 = 0, c_5 = H\}$.

6.1.3. General Statement

We can summarize the above as follows:

THEOREM 6.1. *The normal form Hamiltonian*

$$\mathcal{K} = \mathcal{K}_0 + \varepsilon \mathcal{K}_1 + \frac{\varepsilon^2}{2} \mathcal{K}_2 + \cdots + \frac{\varepsilon^M}{M!} \mathcal{K}_M$$

TABLE II
Poisson Brackets for c_i

$\{ , \}$	c_1	c_2	c_3	c_4	c_5	c_6
c_1	0	$2 c_1$	0	$4 c_2$	0	0
c_2	$-2 c_1$	0	0	$2 c_4$	0	0
c_3	0	0	0	0	0	1
c_4	$-4 c_2$	$-2 c_4$	0	0	0	0
c_5	0	0	0	0	0	0
c_6	0	0	-1	0	0	0

Note. The invariants on the left must be put in the left side of the bracket, whereas the c_i on the top are placed at the right side of the brackets.

constructed through the generating function

$$\mathcal{W} = \mathcal{W}_1 + \varepsilon \mathcal{W}_2 + \frac{\varepsilon^2}{2} \mathcal{W}_3 + \cdots + \frac{\varepsilon^{M-1}}{(M-1)!} \mathcal{W}_M$$

and the initial system $\mathcal{H} = \mathcal{H}_0 + \mathcal{H}_1$ given by

$$\begin{aligned} \mathcal{H}(\ell, g, h, L, G, H) = & -\frac{\mu^2}{2L^2} - \Omega H \\ & + \sum_{\substack{(j, l, m) \in \mathbf{Z}^3 \\ k \in \mathbf{Z}^+ \cup \{0\}}} \left(\frac{r}{\alpha}\right)^j \left(\frac{R}{\beta}\right)^k \mathcal{F}^{(l, m)}(g, h, L, G, H), \end{aligned}$$

where

$$(i) \quad \mathcal{H}_0 = -\frac{\mu^2}{2L^2} - \Omega H \quad \text{or} \quad (ii) \quad \mathcal{H}_0 = -\frac{\mu^2}{2L^2} \quad \text{or} \quad (iii) \quad \mathcal{H}_0 = -\Omega H$$

and

$$\begin{aligned} \mathcal{F}^{(l, m)}(g, h, L, G, H) = & \mathcal{C}^{(l, m)}(L, G, H) \cos(lg + mh) \\ & + \mathcal{S}^{(l, m)}(L, G, H) \sin(lg + mh), \end{aligned}$$

is defined over the reduced phase space, \mathcal{Q} , which is one of the following:

(a) *Regular reduction*, $\mathcal{J} = -\mu^2/(2L^2)$ (cases (i) and (ii)); then for a fixed value of $L > 0$, $\mathcal{Q} = S_L^2 \times S_L^2$ with

$$S_L^2 \times S_L^2 = \{(\mathbf{a}, \mathbf{b}) \in \mathbf{R}^6 \mid a_1^2 + a_2^2 + a_3^2 = L^2, b_1^2 + b_2^2 + b_3^2 = L^2\}.$$

(b) *Singular reduction*, $\mathcal{J} = -\Omega H$ (cases (i) and (iii)); then, for a fixed value of H satisfying $0 \leq |H| \leq G$, $\mathcal{Q} = \mathbf{R}^6/(S^1 \times S^1)_H$ with

$$\mathbf{R}^6/(S^1 \times S^1)_H = \{\mathbf{c} \in \mathbf{R}^6 \mid c_1 c_4 = c_2^2 + c_3^2, c_5 = H\}.$$

Moreover, the normal form \mathcal{K} is defined for all types of bounded orbits as a function of the corresponding invariants; i.e., $\mathcal{K} \equiv \mathcal{K}(\mathbf{a}, \mathbf{b}; L)$ in (a) and $\mathcal{K} \equiv \mathcal{K}(\mathbf{c}; H)$ in (b).

Proof. See Sections 6.1.1 and 6.1.2. ■

6.2. Further Reduction in the Absence of Resonances

In some cases it is possible to further reduce the normal form \mathcal{K} and construct another Hamiltonian of one degree of freedom. This is the so-called integral approximation of \mathcal{K} .

The second reduction can be done whether or not there are resonant terms in the system. If \mathcal{J} (either L or H) is an integral of \mathcal{H}_0 , approximate resonances appear for specific relations between the angles ℓ and h in \mathcal{H}_i , that is, for certain nonnull terms $T_i^{(l,m)} = \sum_{j,k} P_i^{(j,k,l,m)}$ but such that the Poisson brackets $\{\mathcal{J}, T_i^{(l,m)}\}$ are close to zero in a region of Δ_D . In these circumstances, it is not possible to carry out the second reduction in the whole phase space as the dimension of the resonant-space is two. However, it can occur that for a particular model, all the integers j and k related to a term $\exp[\imath(j\ell + kh)]$ —supposing a certain function expanded in Fourier series in ℓ —are such that the linear combination $jn - k\Omega$ does not vanish. (Note that j , k , and Ω are constants but n is a function varying with respect to the time.) Then, a second normal form could be calculated; see some examples in [11, 12, 14, 37, 40]. So, if the first normal form depends on h the second normalisation procedure consists in making H an integral out of it. On the contrary, if the first normal form depends on ℓ the second normal form is built so that the action L becomes an integral of it.

In the case that the two normalisations could be executed, the order of performing the two normal forms does not alter the final result since the following diagram commutes:

$$\begin{array}{ccc} \mathbf{R}^6 & \xrightarrow{L \text{ is an integral}} & S_L^2 \times S_L^2 \\ \downarrow H \text{ is an integral} & & \downarrow H \text{ is an integral} \\ \mathbf{R}^6 / (S^1 \times S^1)_H & \xrightarrow{L \text{ is an integral}} & \mathcal{T}_{L,H} \end{array}$$

Here, $\mathcal{T}_{L,H}$ is going to be the second reduced phase space and we need to describe it. For doing so we repeat the steps given in [12].

From a practical point of view, the second reduction can be performed up to any order for moderate, slow, and fast rotations. The reason is that as the first normal form defines a two-degree-of-freedom system, so L or H becomes an integral; hence $-\mu^2/(2L^2)$ or $-\Omega H$ can be considered as a constant of motion and the corresponding Lie operator needed to perform the second normalisation reduces to, respectively, $\mathcal{L}_{\mathcal{H}_0} = -\Omega \partial(\cdot)/\partial h$ or $\mathcal{L}_{\mathcal{H}_0} = n \partial(\cdot)/\partial \ell$. So, the process to obtain \mathcal{H}_i and \mathcal{W}_i up to any order is the same as those exposed in Section 5.

Now we have to define $\mathcal{T}_{L,H}$ from $S_L^2 \times S_L^2$ although we could do it from the generators of $\mathbf{R}^6 / (S^1 \times S^1)_H$. We start by defining an S^1 -action ϱ on $S_L^2 \times S_L^2$ as

$$\varrho: S^1 \times (S_L^2 \times S_L^2) \rightarrow S_L^2 \times S_L^2 : (h, (\mathbf{a}, \mathbf{b})) \mapsto (R_h \mathbf{a}, R_h \mathbf{b}),$$

where R_h is the matrix given already in (42).

The algebra of polynomials on $S_L^2 \times S_L^2$ invariant under ϱ is generated by

$$\begin{aligned} \pi_1 &= a_1^2 + a_2^2, & \pi_2 &= a_1 b_2 - a_2 b_1, & \pi_3 &= a_3, \\ \pi_4 &= b_1^2 + b_2^2, & \pi_5 &= a_1 b_1 + a_2 b_2, & \pi_6 &= b_3, \end{aligned} \quad (44)$$

together with the constraints

$$\pi_1 + \pi_3^2 = L^2, \quad \pi_4 + \pi_6^2 = L^2, \quad \pi_2^2 + \pi_5^2 = \pi_1 \pi_4. \quad (45)$$

Taking the mapping

$$\pi_H: S_L^2 \times S_L^2 \rightarrow \{H\} \times \mathbf{R}^3: (\mathbf{a}, \mathbf{b}) \mapsto (H, \tau_1, \tau_2, \tau_3) \equiv (H, \boldsymbol{\tau}),$$

where

$$\tau_1 = \frac{1}{2}(\pi_3 - \pi_6), \quad \tau_2 = \pi_2, \quad \tau_3 = \pi_5,$$

we define the invariants τ_1 , τ_2 , and τ_3 in terms of \mathbf{a} and \mathbf{b} as

$$\tau_1 = \frac{1}{2}(a_3 - b_3), \quad \tau_2 = a_1 b_2 - a_2 b_1, \quad \tau_3 = a_1 b_1 + a_2 b_2. \quad (46)$$

As $2G \cos I = a_3 + b_3$ then $H = \frac{1}{2}(a_3 + b_3)$, $H = a_3 - \tau_1$, and $H = \tau_1 - b_3$. Note that $\boldsymbol{\tau}$ may be expressed in Delaunay variables through formulæ (38).

The constraints (45) are used to define the corresponding phase space. This space, $\mathcal{T}_{L,H}$, is defined as the image of $S_L^2 \times S_L^2$ by π_H ; that is,

$$\begin{aligned} \mathcal{T}_{L,H} &= \pi_H(S_L^2 \times S_L^2) \\ &= \{\boldsymbol{\tau} \in \mathbf{R}^3 \mid \tau_2^2 + \tau_3^2 = [L^2 - (\tau_1 - H)^2][L^2 - (\tau_1 + H)^2]\}, \end{aligned} \quad (47)$$

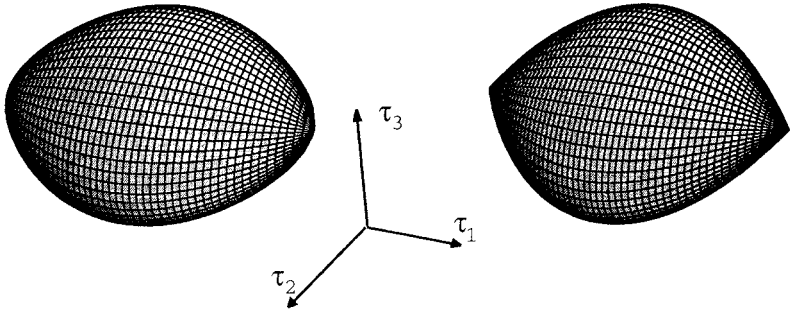


FIG. 1. Two plots of the phase space $\mathcal{T}_{L,H}$. On the left is a smooth surface of revolution which corresponds to the case $H/L = 2/11$. The right corresponds to the case $L > 0$ and $H \equiv 0$. It is a singular surface of revolution with two singularities at the extreme points $(\pm L, 0, 0)$.

for $0 \leq |H| \leq L$ and $L > 0$. Note that τ_2 and τ_3 always lie in the interval $[H^2 - L^2, L^2 - H^2]$ whereas τ_1 belongs to $[H - L, L - H]$.

Cushman proved that when $0 < |H| < L$, $\mathcal{T}_{L,H}$ is diffeomorphic to a two-sphere S^2 and therefore the reduction is regular in that region of the phase space. However, when $H = 0$ then $\mathcal{T}_{L,0}$ is a topological two-sphere with two singular points: the vertices at $(\pm L, 0, 0)$. The reason for the existence of these two points is that the S^1 -action ϱ has two fixed points, $L(\pm 1, 0, 0, \mp 1, 0, 0)$, and consequently ϱ is not free. Finally, when $|H| = L$ the phase space $\mathcal{T}_{\pm L,L}$ gets reduced to a point. See Fig. 1 for two representations of the phase space $\mathcal{T}_{L,H}$.

It is not difficult to prove that g and G can be expressed in terms of τ . We have:

$$\begin{aligned} 2G^2 &= L^2 + H^2 - \tau_1^2 + \tau_3, \\ \cos g &= \frac{-\tau_2}{\sqrt{(L^2 - H^2)^2 - (\tau_1^2 - \tau_3)^2}}, \\ \sin g &= \tau_1 \sqrt{\frac{2(L^2 + H^2 - \tau_1^2 + \tau_3)}{(L^2 - H^2)^2 - (\tau_1^2 - \tau_3)^2}}. \end{aligned}$$

With these relations it is possible to express the quantities $\sin I$, $\cos I$, $\sin g$, $\cos g$, and G in terms of τ , L and H . Besides, other variables such as e and η can be put in terms of the invariants L and H through the variable G .

Rectilinear orbits must satisfy $G = H = 0$. Taking also into account the constraint appearing in (47), we know that they are defined on the one-dimensional set: $\mathcal{R}_{L,0} = \{\tau \in \mathbf{R}^3 \mid \tau_2 = 0, \tau_3 = \tau_1^2 - L^2\}$. Thus, excepting orbits with $\|\mathbf{x}\| = 0$ we could analyse rectilinear trajectories. Circular orbits are concentrated in a unique point of $\mathcal{T}_{L,H}$ with coordinates $(0, 0, L^2 - H^2)$ whereas equatorial trajectories in this double-reduced phase space are represented in the negative extreme point of $\mathcal{T}_{L,H}$ with coordinates $(0, 0, H^2 - L^2)$.

TABLE III
Poisson Brackets for the τ_i

$\{, \}$	τ_1	τ_2	τ_3
τ_1	0	$2\tau_3$	$-2\tau_2$
τ_2	$-2\tau_3$	0	$-4\tau_1 (\tau_1^2 - L^2 - H^2)$
τ_3	$2\tau_2$	$4\tau_1 (\tau_1^2 - L^2 - H^2)$	0

Note. The invariants on the left must be put in the left side of the bracket, whereas the τ_i on the top are placed at the right side of the brackets.

This time the second normal form is represented by a Hamiltonian \mathcal{Z} expressed in terms of the τ_i . It defines a one-degree-of-freedom system with L and H as its integrals. The Poisson brackets of the τ are in Table III.

In the following we summarize the contents of Section 6.2.

THEOREM 6.2. *The normal form Hamiltonian*

$$\mathcal{Z} = \mathcal{Z}_0 + \varepsilon \mathcal{Z}_1 + \frac{\varepsilon^2}{2} \mathcal{Z}_2 + \cdots + \frac{\varepsilon^M}{M!} \mathcal{Z}_M$$

constructed through two successive generating functions

$$\mathcal{W}^{(i)} = \mathcal{W}_1^{(i)} + \varepsilon \mathcal{W}_2^{(i)} + \frac{\varepsilon^2}{2} \mathcal{W}_3^{(i)} + \cdots + \frac{\varepsilon^{M-1}}{(M-1)!} \mathcal{W}_M^{(i)} \quad \text{with } i = 1, 2$$

from the system $\mathcal{H} = \mathcal{H}_0 + \mathcal{H}_1$ given by

$$\begin{aligned} \mathcal{H}(\ell, g, h, L, G, H) = & -\frac{\mu^2}{2L^2} - \Omega H \\ & + \sum_{\substack{(j,l,m) \in \mathbb{Z}^3 \\ k \in \mathbb{Z}^+ \cup \{0\}}} \left(\frac{r}{\alpha}\right)^j \left(\frac{R}{\beta}\right)^k \mathcal{F}^{(l,m)}(g, h, L, G, H), \end{aligned}$$

where

$$(i) \quad \mathcal{H}_0 = -\frac{\mu^2}{2L^2} - \Omega H \quad \text{or} \quad (ii) \quad \mathcal{H}_0 = -\frac{\mu^2}{2L^2}, \quad \text{or} \quad (iii) \quad \mathcal{H}_0 = -\Omega H$$

and

$$\begin{aligned} \mathcal{F}^{(l,m)}(g, h, L, G, H) = & \mathcal{C}^{(l,m)}(L, G, H) \cos(lg + mh) \\ & + \mathcal{S}^{(l,m)}(L, G, H) \sin(lg + mh), \end{aligned}$$

satisfies the following properties:

$$(a) \quad \mathcal{Z}_0 \equiv \mathcal{H}_0.$$

(b) \mathcal{Z} is built in two steps. First \mathcal{K} and $\mathcal{W}^{(1)}$ are determined from \mathcal{H} as it is explained in Theorem 6.1. Thus either $\mathcal{I}_1 = -\mu^2/(2L^2)$ (cases (i) and (ii)), or $\mathcal{I}_1 = -\Omega H$ (cases (i) and (iii)) becomes an integral—the first formal integral—of \mathcal{K} . In a second step \mathcal{Z} and $\mathcal{W}^{(2)}$ are calculated from \mathcal{K} using $\mathcal{I}_2 = -\Omega H$ (cases (i) and (iii)), or $\mathcal{I}_2 = -\mu^2/(2L^2)$ (cases (i) and (ii)). Then, \mathcal{I}_2 becomes an integral—the second formal integral—of \mathcal{Z} .

(c) \mathcal{Z} is defined over the phase space (second reduced phase space) $\mathcal{T}_{L,H}$, a two-dimensional space embedded in \mathbf{R}^3 such that:

$$\mathcal{T}_{L,H} = \{\tau \in \mathbf{R}^3 \mid \tau_2^2 + \tau_3^2 = [L^2 - (\tau_1 - H)^2][L^2 - (\tau_1 + H)^2]\}.$$

Moreover, the normal form $\mathcal{Z}(\tau; L, H)$ is defined for all types of bounded orbits. In general, the reduction is singular.

Proof. See the preceding paragraphs. ■

7. HYDROGEN ATOM IN ORTHOGONAL ELECTRIC AND MAGNETIC FIELDS

We illustrate the theory exposed in the previous sections with a problem of classical mechanics. Nevertheless, we do not intend to study the resulting normal forms.

In the presence of constant orthogonal magnetic and electric fields, the Hamiltonian function of the hydrogen atom is

$$\mathcal{H}(\mathbf{x}, \mathbf{X}) = \frac{1}{2} \mathbf{X} \cdot \mathbf{X} - \frac{C}{\|\mathbf{x}\|} + Fx + \frac{1}{2} B(xY - yX) + \frac{1}{8} B^2(x^2 + y^2), \quad (48)$$

where the directions of the magnetic and electric fields are, respectively, the axes z and x . The term Fx is the electrostatic potential describing the Stark effect (F has dimension [length/time²]) while the terms having the constant B of physical dimension [1/time] refer to the linear and quadratic Zeeman effect. Besides, $C > 0$ represents the effective charge of the potential. The physical dimension of C is [length³/time²].

Hamiltonian systems of this type have been studied in [14] in order to analyse the presence of monodromy. The rotating Stark effect can be represented by a Hamiltonian such as (48) with a term proportional to $xY - yX$ and the other proportional to x , i.e. a system such as (48) but without taking into account the quadratic term of the Zeeman effect. This system was analysed in [18]. Gutzwiller [25] dealt with the diamagnetic Kepler problem, which corresponds to a Hamiltonian with the action of a uniform magnetic field in the direction z . This model is represented by a Hamiltonian of the type (48) but without the electric component.

After making the changes $C \equiv \mu$ and $B \equiv -2\Omega$, Hamiltonian (48) can be written as a combination of polar-nodal and Delaunay variables as:

$$\mathcal{H}(r, f, g, h, L, G, H) = -\frac{\mu^2}{2L^2} + Fx - \Omega H + \frac{1}{2} \Omega^2(x^2 + y^2), \quad (49)$$

where x and y are replaced using (4) and (5). Thus we obtain:

$$x = r [\cos(f+g) \cos h - \sin(f+g) \sin h \cos I],$$

$$x^2 + y^2 = \frac{1}{2} r^2 \{2 - \sin^2 I + \sin^2 I \cos[2(f+g)]\}.$$

Different problems arise according to the suitable scalings of \mathcal{H} , i.e., depending on the relative values of C , B , and F .

As a first approach we take the case for which the Keplerian part and the rotating term (Coriolis-like term) are of the same order. It is the situation of a strong magnetic field (or, according to how we quoted it in last sections, moderate rotations). An appropriate scaling in this case is:

$$\mathcal{H}_0(L, H) = -\frac{\mu^2}{2L^2} - \Omega H, \quad \mathcal{H}_1(r, f, g, h, L, G, H) = x + \frac{\Omega^2}{2F} (x^2 + y^2).$$

Thus, $\mu^2/(2L^2) \approx \Omega |H|$; that is, $\Omega |H|/L \approx \frac{1}{2}n$. Besides, the small parameter ε is taken equal to F and then $\mathcal{H} = \mathcal{H}_0 + \varepsilon \mathcal{H}_1$. Notice that the Stark effect is of the same size as the quadratic part of the Zeeman effect.

The reduction process consists in calculating first the normal form, deriving then the reduced phase space, and putting the normalised Hamiltonian as a function of the generator of this phase space. So, two possibilities are in order: either L or H are used to become integrals of the normal form.

The construction of \mathcal{K} is similar in the two cases. To compute \mathcal{K} and \mathcal{W} we need previously to write \mathcal{H}_1 in the form (14). Then, by means of formulæ (6) and (8) and using that $\vartheta = f + g$, the terms $\cos f$ and $\sin f$ have to be put in terms of nonnegative powers of R and integer powers of the coordinate r . Then we have to take \mathcal{K}_1 as one of the two averages: $\mathcal{K}_1 = \frac{1}{2\pi} \int_0^{2\pi} \mathcal{H}_1 d\ell$ or $\mathcal{K}_1 = \frac{1}{2\pi} \int_0^{2\pi} \mathcal{H}_1 dh$. In the first case the result expressed in the (transformed) Delaunay variables, after dropping the primes, is a sum in the angles g and h . Concretely, for the first case:

$$\mathcal{K}_1 = -\frac{3eL^2}{4\mu} [(1 - \cos I) \cos(g-h) + (1 + \cos I) \cos(g+h)]$$

$$+ \frac{\Omega^2 L^4}{8F\mu^2} [6e^2 - 2 \sin^2 I - 3e^2 \sin^2 I + 5e^2 \sin^2 I \cos(2g)],$$

whereas for the second:

$$\mathcal{K}_1 = \frac{\Omega^2}{4F} r^2 \{2 - \sin^2 I + \sin^2 I \cos[2(f+g)]\}.$$

The corresponding generating functions $\mathcal{W} \equiv \mathcal{W}_1$ are written in Delaunay variables. Besides, to make all the computations valid for any value of $e < 1$, $I_0^{(b,m)}$ is used as all the terms in $\tilde{\mathcal{H}}_1$ satisfy $j+1 \geq k$. We do not write explicitly the generating functions as they are quite big, but they are analytic with respect to all their arguments. We stress the convenience of using closed expressions for \mathcal{W} as they are valid for all values lying in the domain Δ_D . Second order cannot be reached straightforwardly as $\tilde{\mathcal{H}}_2$ contains terms of the type defined in (16).

Normal forms up to first order can be put in terms of their corresponding invariants. For the first case (L becomes an integral) it is given by

$$\mathcal{K}_0(\mathbf{a}, \mathbf{b}; L) = -\frac{\mu^2}{2L^2} - \frac{\Omega}{2}(a_3 + b_3),$$

and the first order is given by the polynomial:

$$\mathcal{K}_1(\mathbf{a}, \mathbf{b}; L) = \frac{3L}{4\mu}(b_1 - a_1) + \frac{\Omega^2 L^2}{4F\mu^2}[(a_1 - b_1)^2 + (a_2 - b_2)^2 + a_3 b_3]. \quad (50)$$

Hamiltonian $\mathcal{K}(\mathbf{a}, \mathbf{b}; L)$ defines a two-degree-of-freedom system in $S_L^2 \times S_L^2$ with (internal) parameter L and external parameters μ , Ω and F . The use of invariants is advisable when analysing \mathcal{K} as we can consider circular and equatorial trajectories.

Now, for the second case (H becomes an integral), we need to put L in terms of the invariants \mathbf{c} defined in (40). We arrive at the identity:

$$-\frac{\mu^2}{2L^2} = \frac{1}{2}(c_4 + c_6^2) - \frac{\mu}{\sqrt{c_1 + c_3^2}}.$$

Hence, the normalised Hamiltonian is given by:

$$\mathcal{K}_0(\mathbf{c}; H) = \frac{1}{2}(c_4 + c_6^2) - \frac{\mu}{\sqrt{c_1 + c_3^2}} - \Omega H, \quad \mathcal{K}_1(\mathbf{c}; H) = \frac{\Omega^2}{4F} c_1.$$

Hamiltonian $\mathcal{K}_1(\mathbf{c}; H)$ defines a two-degree-of-freedom system in $\mathbf{R}^6/(S^1 \times S^1)_H$ with external parameters Ω and F . Note that since \mathcal{K} is written in invariants we can consider rectilinear and equatorial trajectories.

If there are no resonant terms in the perturbation, i.e., if, after developing \mathcal{H}_1 in Fourier series of ℓ up to a certain order, all the expressions $\exp[i(j\ell + kh)]$ are such that the linear combinations $|jn - k\Omega|$ (accordingly to the initial value $L(t_0)$) remain strictly positive, and taking into account the values of the constants Ω and F , we can perform a second reduction and reduce \mathcal{H}_1 . To this end one can transform either $\mathcal{K}_1(\mathbf{a}, \mathbf{b}; L)$ or

$\mathcal{H}_1(\mathbf{c}; H)$, obtaining the same result. The unperturbed Hamiltonian is now $\mathcal{H}_0(\tau; L, H) = -\mu^2/(2L^2) - \Omega H$. The second reduced Hamiltonian is

$$\mathcal{H}_1(\tau; L, H) = -\frac{\Omega^2 L^2}{4F\mu^2} (3\tau_1^2 + 2\tau_3), \quad (51)$$

and it is defined on the surface \mathcal{T}_{LH} for all values $0 \leq |H| \leq L$ and $L > 0$. So, all type of trajectories (including rectilinear) can be analysed in \mathcal{T}_{LH} .

A second possibility refers to weak magnetic perturbations, e.g., slow rotations. Then $\Omega |H|/L \ll \frac{1}{2}n$ and the corresponding scalings are either

$$\mathcal{H}_0(L) = -\frac{\mu^2}{2L^2}, \quad (52)$$

$$\mathcal{H}_1(r, f, g, h, L, G, H) = -n_0 H + \frac{n_0}{\Omega} \left[Fx + \frac{1}{2} \Omega^2 (x^2 + y^2) \right],$$

which must be used when $\Omega |H| \approx |Fx + \frac{1}{2} \Omega^2 (x^2 + y^2)|$, or

$$\mathcal{H}_0(L) = -\frac{\mu^2}{2L^2}, \quad \mathcal{H}_1(H) = -n_0 H, \quad (53)$$

$$\mathcal{H}_2(r, f, g, h, L, G, H) = \frac{2n_0^2}{\Omega^2} \left[Fx + \frac{1}{2} \Omega^2 (x^2 + y^2) \right],$$

which must be used when $\Omega |H| \gg |Fx + \frac{1}{2} \Omega^2 (x^2 + y^2)|$. This is the case studied in [14]. The small parameter is $\varepsilon = \Omega/n_0$ and n_0 is a constant representing mean motion for the initial conditions, e.g., $n_0 = \mu^2/L(t_0)^3$.

In both cases (52) and (53) the normalisation consists in calculating a new Hamiltonian for which L becomes an integral. On this occasion, the process can be pushed to any order as no limitation occurs due to the appearance of $\mathcal{L}_{\mathcal{H}_0}$. The generating functions can be calculated in closed form and no use of $I_0^{(b,m)}$ is made.

The first-order normal form of (52) is

$$\mathcal{H}'_1(\mathbf{a}, \mathbf{b}; L) = -\frac{n_0}{2} (a_3 + b_3) + \frac{n_0 F}{\Omega} \mathcal{H}_1(\mathbf{a}, \mathbf{b}; L),$$

whereas, for (53) we have, up to order two:

$$\mathcal{H}'_1(\mathbf{a}, \mathbf{b}; L) = -\frac{n_0}{2} (a_3 + b_3), \quad \mathcal{H}'_2(\mathbf{a}, \mathbf{b}; L) = \frac{2n_0^2 F}{\Omega^2} \mathcal{H}_1(\mathbf{a}, \mathbf{b}; L).$$

In both cases, $\mathcal{H}_1(\mathbf{a}, \mathbf{b}; L)$ was given by (50) and the normal forms define a dynamical system with two degrees of freedom in $S_L^2 \times S_L^2$. Besides, $\mathcal{H}'_0(\mathbf{a}, \mathbf{b}; L) = -\mu^2/(2L^2)$.

If there is no resonance due to the special aspect of the perturbation, it is possible to perform a second reduction. This reduction can be executed to any order in (52) and (53). Thence, the second order form for (53) is

$$\mathcal{Z}'_1(\tau; L, H) = -n_0 H, \quad \mathcal{Z}'_2(\tau; L, H) = \frac{2n_0^2 F}{\Omega^2} \mathcal{Z}_1(\tau; L, H),$$

where $\mathcal{Z}_1(\tau; L, H)$ is given by (51). The normal form defines in this case a system of one degree of freedom in \mathcal{T}_{LH} . Now, $\mathcal{Z}'_0(\tau; L, H) = -\mu^2/(2L^2)$.

Finally we analyse the possibility of very strong magnetic fields for which $\frac{1}{2}n \ll \Omega |H|/L$ (i.e., fast rotations). Then, two correct scalings are

$$\mathcal{H}_0(H) = -\Omega H,$$

$$\mathcal{H}_1(r, f, g, h, L, G, H) = -\frac{\mu^2 \Omega}{2L^2 n_0} + \frac{\Omega}{n_0} \left[Fx + \frac{1}{2} \Omega^2 (x^2 + y^2) \right], \quad (54)$$

when $\mu^2/(2L^2) \approx |Fx + \frac{1}{2} \Omega^2 (x^2 + y^2)|$ and

$$\mathcal{H}_0(H) = -\Omega H, \quad \mathcal{H}_1(L) = -\frac{\mu^2 \Omega}{2L^2 n_0}, \quad (55)$$

$$\mathcal{H}_2(r, f, g, h, L, G, H) = \frac{2\Omega^2}{n_0^2} \left[Fx + \frac{1}{2} \Omega^2 (x^2 + y^2) \right],$$

if $\mu^2/(2L^2) \gg |Fx + \frac{1}{2} \Omega^2 (x^2 + y^2)|$. The small parameter is $\varepsilon = n_0/\Omega$.

Normal forms for (54) and (55) are computed so that H becomes an integral. The process can be pushed to any order due to the form of $\mathcal{L}_{\mathcal{H}_0}$. The generating functions are computed in closed form without using $I_0^{(b,m)}$.

Now, the first-order normal form for (54) is

$$\mathcal{K}'_1(\mathbf{c}; H) = \frac{\Omega}{n_0} \left[\frac{1}{2} (c_4 + c_6^2) - \frac{\mu}{\sqrt{c_1 + c_3^2}} \right] + \frac{\Omega F}{n_0} \mathcal{K}_1(\mathbf{c}; H),$$

whereas, for (55) we have, up to order two:

$$\mathcal{K}'_1(\mathbf{c}; H) = \frac{\Omega}{n_0} \left[\frac{1}{2} (c_4 + c_6^2) - \frac{\mu}{\sqrt{c_1 + c_3^2}} \right], \quad \mathcal{K}'_2(\mathbf{c}; H) = \frac{2\Omega^2 F}{n_0^2} \mathcal{K}_1(\mathbf{c}; H).$$

In both cases, $\mathcal{K}_1(\mathbf{c}; H)$ was given by (50) and the normal forms define a dynamical system with two degrees of freedom in $\mathbf{R}^6/(S^1 \times S^1)_H$.

For nonresonant perturbations, we could perform a second reduction to any order in (54) and (55). The second-order normal form for (55) is

$$\mathcal{Z}'_1(\tau; L, H) = -\frac{\mu^2 \Omega}{2L^2 n_0}, \quad \mathcal{Z}'_2(\tau; L, H) = \frac{2\Omega^2 F}{n_0^2} \mathcal{Z}_1(\tau; L, H),$$

where $\mathcal{L}_1(\tau; L, H)$ is given by (51). The normal form defines a dynamical system of one degree of freedom in \mathcal{T}_{LH} . Besides, $\mathcal{L}'_0(\tau; L, H) = -\Omega H$.

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REFERENCES

1. R. Abraham and J. E. Marsden, "Foundations of Mechanics," Dover, Redwood City, CA, 1985.
2. M. Abramowitz and I. A. Stegun, "Handbook of Mathematical Functions," Dover, New York, 1965.
3. J. M. Arms, R. H. Cushman, and M. J. Gotay, A universal reduction procedure for Hamiltonian group actions, in "The Geometry of Hamiltonian Systems" (T. Ratiu, Ed.), M. R. S. I. Workshop Proceedings, pp. 33–51, Springer-Verlag, Berlin/New York, 1991.
4. G. D. Birkhoff, "Dynamical Systems," A.M.S. Coll. Publications 9, American Mathematical Society, Providence, RI, 1927.
5. D. Brouwer and G. M. Clemence, "Methods of Celestial Mechanics," Academic Press, New York/London, 1961.
6. V. A. Brumberg, "Analytical Techniques of Celestial Mechanics," Springer-Verlag, Berlin/Heidelberg, 1995.
7. M. A. Chaudhry and S. M. Zubair, Two integrals arising in generalized inverse Gaussian model and heat conduction problems, *SIAM Rev. Problem 92–15* **34** (1992), 498; comments by M. L. Glasser, *SIAM Rev.* **35** (1993), 503–504 and by H. J. Haubold and A. M. Mathai, *SIAM Rev.* **40** (1998), 995–997.
8. M. A. Chaudhry and S. M. Zubair, Generalized incomplete gamma functions with applications, *J. Comput. Appl. Math.* **55** (1994), 99–124.
9. M. A. Chaudhry, N. M. Temme, and E. J. M. Velting, Asymptotics and closed form of a generalized incomplete gamma function, *J. Comput. Appl. Math.* **67** (1996), 371–379.
10. R. C. Churchill, M. Kummer, and D. L. Rod, On averaging, reduction, and symmetry in Hamiltonian systems, *J. Differential Equations* **49** (1983), 359–414.
11. S. L. Coffey and A. Deprit, Third order solution to the main problem in satellite theory, *J. Guid. Control and Dynam.* **5** (1982), 363–371.
12. R. H. Cushman, Reduction, Brouwer's Hamiltonian, and the critical inclination, *Celestial Mech.* **31** (1983), 401–429.
13. R. H. Cushman, Normal form for Hamiltonian vectorfields with periodic flow, in "Differential Geometric Methods in Mathematical Physics" (S. Sternberg, Ed.), pp. 125–144, D. Reidel, Dordrecht, 1984.
14. R. H. Cushman and D. A. Sadovskii, Monodromy in the hydrogen atom in crossed fields, *Phys. D* **142** (2000), 166–196.
15. A. Deprit, Canonical transformations depending on a small parameter, *Celestial Mech.* **1** (1969), 12–30.
16. A. Deprit, The elimination of the parallax in satellite theory, *Celestial Mech.* **24** (1981), 111–153.

17. A. Deprit, Delaunay normalisations, *Celestial Mech.* **26** (1982), 9–21.
18. A. Deprit, Dynamics of orbiting dust under radiation pressure, in “The Big Bang and Georges Lemaitre” (A. Berger, Ed.), pp. 151–180, D. Reidel, Dordrecht, 1984.
19. A. Deprit, J. Henrard, and A. Rom, Lunar ephemeris: Delaunay’s theory revisited, *Science* **168** (1970), 1569–1570.
20. A. Deprit, J. Henrard, and A. Rom, Analytical lunar ephemeris: Delaunay’s theory, *Astronom. J.* **76** (1971), 269–272.
21. A. Deprit, J. Palacián, and E. Deprit, The relegation algorithm, *Celestial Mech. Dynam. Astronom.* **79** (2001), 157–182.
22. A. Elife and S. Ferrer, Reductions, relative equilibria and bifurcations in the generalized van der Waals potential: relation to the integrable cases, *Phys. Rev. Lett.* **72** (1994), 985–988.
23. A. Elife, J. Palacián and M. Arribas, Relegation of the proper rotation in attitude dynamics of a spacecraft, in “Proceedings of the IV ESA Symposium on Spacecraft Flight Dynamics, ESOC, Darmstadt, 1991,” pp. 55–59.
24. A. Erdélyi, W. Magnus, F. Oberhettinger, and F. G. Tricomi, “Higher Transcendental Functions,” Vol. II, McGraw–Hill, New York, 1953.
25. M. C. Gutzwiller, “Chaos in Classical and Quantum Mechanics,” Interdisciplinary Applied Mathematics, Vol. 1, Springer-Verlag, New York, 1990.
26. M. Kummer, On the 3-dimensional lunar problem and other perturbations of the Kepler problem, *J. Math. Anal. Appl.* **101** (1983), 142–194.
27. M. Kummer, A group theoretical approach to a certain class of perturbations of the Kepler problem, *Arch. Rational Mech. Anal.* **91** (1985), 55–82.
28. P. Kustaanheimo and E. Stiefel, Perturbation theory of Kepler motion based on spinor regularization, *J. Reine Angew. Math.* **219** (1965), 204–219.
29. T. Levi-Civita, Sur la régularisation du problème des trois corp, *Acta Math.* **42** (1920), 99–144.
30. J. Marsden and A. Weinstein, Reduction of symplectic manifolds with symmetries, *Rep. Math. Phys.* **5** (1974), 121–130.
31. K. R. Meyer, Symmetries and integrals in mechanics, in “Dynamical Systems” (M. M. Peixoto, Ed.), pp. 259–272, Academic Press, New York, 1973.
32. K. R. Meyer, Scaling Hamiltonian systems, *SIAM J. Math. Anal.* **15** (1984), 877–889.
33. K. R. Meyer, A Lie transform tutorial II, in “Computer Aided Proofs in Analysis” (K. R. Meyer and D. S. Schmidt, Eds.), IMA Volumes in Mathematics and Its Applications, Vol. 28, 190–210, Springer-Verlag, New York, 1991.
34. K. R. Meyer and G. R. Hall, “Introduction to Hamiltonian Dynamical Systems and the N -Body Problem,” Applied Mathematical Sciences, Vol. 90, Springer-Verlag, New York, 1992.
35. J. Moser, Regularization of Kepler’s problem and the averaging method on a manifold, *Comm. Pure Appl. Math.* **23** (1970), 609–636.
36. C. Osácar and J. Palacián, Decomposition of functions for elliptic orbits, *Celestial Mech. Dynam. Astronom.* **60** (1994), 207–223.
37. J. Palacián, “Teoría del satélite artificial: armónicos teserales y su relegación mediante simplificaciones algebraicas,” Ph.D. thesis (1992), Publicaciones del Seminario Matemático García de Galdeano, Serie II, Sección 2 45, Universidad de Zaragoza, Zaragoza, 1993.
38. J. Palacián, Closed-form normalisations of perturbed two-body problems, *Chaos, Solitons Fractals* **13** (2002), 853–874.
39. J. Palacián, Normalisations in the theory of artificial satellites taking into account the tesseral harmonic coefficients, preprint, 2001.
40. J. Palacián, J. F. San Juan, and P. Yanguas, Analytical theory for the SPOT satellite, *Adv. Astron. Sci.* **95** (1997), 375–382.

41. J. Palacián and P. Yanguas, Simplification of perturbed Hamiltonians through Lie transformations, in "Hamiltonian Systems and Celestial Mechanics (HAMYS-98)" (J. Delgado, E. A. Lacomba, E. Pérez-Chavela, and J. Llibre, Eds.), World Scientific, Singapore, 2000.
42. W. M. Smart, "Celestial Mechanics," Longmans, Green, London, 1953.
43. F. Tisserand, "Traité de mécanique céleste," Vol. I, Gauthiers-Villars, Paris, 1889.
44. E. T. Whittaker, "A Treatise on the Analytical Dynamics of Particles and Rigid Bodies," Cambridge University Press, Cambridge, MA, 1927.
45. S. Wolfram, "The Mathematica Book," 4th ed., Wolfram Media/Cambridge University Press, Cambridge, MA, 1999.
46. P. Yanguas, "Integrability, Normalization and Symmetries of Hamiltonian Systems in 1-1-1 Resonance," Ph.D. thesis, Departamento de Matemática e Informática, Universidad Pública de Navarra, Pamplona, 1998.
47. P. Yanguas, Perturbations of the isochrone model, *Nonlinearity* **14** (2001), 1–34.